ACT II. SCENE 2: ACHIEVING SOME MEASURE OF CLOSURE

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1. Introduction

We concluded Act I with the so-called "closure problem": how to determine the quantities in the three compartments labelled, electromagnetic, radiative and thermodynamic $\{\delta, \mathbf{J} \mid \chi_{\nu}, \eta_{\nu} \mid p, T, \sigma, \kappa\}$. We also indicated that the first method of choice is to attempt to ascertain these relationships in the comoving frame of the material, and then to Lorentz Transform these results to the laboratory frame.

While our previous Scenes all dealt with matters which progressed fairly logically and contained little in the way of surprises, or "aha!" moments, the closure aspect of our problem is very much based on intuition, insight, and a great deal of artistic license. Sometimes, seemingly silly, or hopelessly simplistic, prescriptions will give the right results. And other times the most careful and cunningly designed closure schemes end in abject failure. It's definitely live and learn.

Our program is to treat each compartment separately and share with you some of the more useful and interesting closure schemes, along with some motivation as to when and where they might be expected to work, and *not* work for you.

2. Electrodynamics: MHD, EHD and Ohm's Law

The Lorentz Transformations between a frame at rest (unprimed) and one moving relative to this frame at a constant velocity \mathbf{u} (the comoving primed frame) for the electromagnetic fields are:

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}$$
$$\mathbf{B}' = \mathbf{B} - \frac{1}{c}\mathbf{u} \times \mathbf{E}$$

correct to order u/c. In fact, in what follows we shall *only* retain terms at this order in u/c, therefore the Lorentz Factor, $\sqrt{1 - u^2/c^2} \approx 1$, from here on.

These forms result from the fact that relativistically, the electric and magnetic fields combine to form the various components of a 4-tensor, whereas the charge density δ and the electric current density **J** are components of a 4-vector. To the same order in u/c, they transform according to:

$$\mathbf{J}' = \mathbf{J} - \delta \mathbf{u}$$

 $\delta' = \delta - \frac{1}{c^2} \mathbf{u} \cdot \mathbf{J}$

A constituitive relation, the so-called Ohm's Law, reckoned in the comoving frame of the fluid is necessary to express the electric current density in terms of the remaining quantities in Maxwell's Equations. For simplicity, we adopt the simple scalar expression

$$\mathbf{J}' = \sigma \mathbf{E}'$$

where the electric conductivity σ has dimensions of sec⁻¹.

This prescription generally "works" when collisions are frequent between the constituents of the gas. Frequent compared to what? Well certainly any dynamical time scale associated with the macroscopic flows. But also frequent compared to the inverse of the electron gyrofrequency: $e|\mathbf{B}|/m_e c$, and the plasma frequency $\sqrt{4\pi N^e c^2/m_e}$. In more general situations, when the gyrofrequency constraint is not satisfied, σ is a tensor (not to be confused with the viscous stress tensor) and the current density can also depend upon the magnetic field **B**'. In situations where electrons and ions do not collide frequently enough (owing to their disparate masses), although they run into their own kind with sufficient rapidity, more modifications are required to Ohm's Law. To be entirely consistent with our notation, we should probably place a prime on σ to indicate that it is a comoving frame quantity—but we won't.

In terms of quantities evaluated in the rest frame of the fluid, this expression reads

$$\mathbf{J} = \delta \mathbf{u} + \sigma \left(\mathbf{E} + rac{1}{c} \mathbf{u} imes \mathbf{B}
ight) \; ,$$

correct to order u/c. In the limit that $\sigma \to \infty$ we must have

$$\mathbf{E} \to -\frac{1}{c}\mathbf{u} \times \mathbf{B} + O(\sigma^{-1})$$

and we must look elsewhere (i.e., Ampère's Equation) to obtain information about **J**. In the opposite limit $\sigma \to 0$ we find

$$\mathbf{J} \to \delta \mathbf{u} + O(\sigma)$$

and we must look elsewhere (i.e., Faraday's Equation) to find the scaling between ${\bf E}$ and ${\bf B}.$

Making no assumption about the size of σ , the divergence of Ampère's Equation

$$c\nabla \times \mathbf{B} = 4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}$$

combined with Gauss's Law

$$\nabla \cdot \mathbf{E} = 4\pi \delta$$

results in the conservation of charge:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \left(\delta \mathbf{u} + \sigma \left[\mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} \right] \right) = 0 \; .$$

And the curl of Ampère's Equation provides an evolution equation for the magnetic field

$$\frac{1}{c}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla \times \left(4\pi\delta \mathbf{u} - c\nabla \times \mathbf{B} + 4\pi\sigma \left[\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}\right]\right) = 0 ,$$

where Faraday's Law has been used to eliminate the curl of the electric field.

If the conductivity is a function of position, then these two evolution equation must be supplemented by Gauss's and Faraday's Laws to determine the solenoidal and irrotational components of **E**. Otherwise, if the conductivity can be taken as a constant, these two equations may be reduced to,

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \delta \mathbf{u} + 4\pi\sigma\delta = -\frac{\sigma}{c}\nabla \cdot (\mathbf{u} \times \mathbf{B})$$

and

$$\frac{1}{c} \left(\frac{\partial}{\partial t} + 4\pi\sigma \right) \frac{\partial \mathbf{B}}{\partial t} - \nabla \times \left(\frac{4\pi\sigma}{c} \mathbf{u} \times \mathbf{B} - c\nabla \times \mathbf{B} \right) = 4\pi\nabla \times \delta \mathbf{u} ,$$

a coupled pair of evolution equations for the magnetic field and the electric charge density. In deriving these equations we have assumed only that $u^2/c^2 \ll 1$, the scalar Ohm's law holds in the rest frame of the fluid, and the electrical conductivity is not a function of position or time. The limiting behavior of this system of equations depends critically upon the size of the electrical conductivity, σ , relative to the inverse of the characteristic fluid time scale $\tau \equiv L/U$, where $\nabla \cdot \mathbf{u} \approx U/L$,

In the zero conductivity limit, for which $\sigma \ll U/L$, this system of equations reduces to the electromagnetic wave equation,

$$\frac{1}{c}\frac{\partial^2 \mathbf{B}}{\partial t^2} - c\nabla^2 \mathbf{B} = 4\pi\nabla \times \delta \mathbf{u}$$

with the advective motion of the electric charge density (equivalently the current \mathbf{J}) as a source term. Where the source term is non zero—usually termed the near field induction zone— it is balanced by the Laplacian of the magnetic field, so

$$|\mathbf{B}| \approx \frac{UL\delta}{c} \approx \frac{U}{c} |\mathbf{E}| \ll |\mathbf{E}|.$$

Where the source terms vanishes—in the far field—the spatial variation of **B** now becomes the electromagnetic wavelength $c\tau$ (not L), so there is a balance between the Laplacian and the second time derivative. The remaining equation simply expresses the conservation of electric charge,

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \delta \mathbf{u} = 0 \; .$$

This limit is sometimes referred to as ideal *Electrohydrodynamics*, or EHD, and arises in solid state physics, but almost never in astrophysics.

In the opposite limit of *infinite conductivity*, for which $\sigma \gg c^2/UL$, we obtain the magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \; ,$$

which indicates that the magnetic field is frozen into the fluid. The remaining equation is simply a statement of the electric charge density in the laboratory frame:

$$\delta = -\frac{1}{4\pi c} \nabla \cdot (\mathbf{u} \times \mathbf{B}) = \frac{1}{4\pi} \nabla \cdot \mathbf{E} = \frac{1}{c^2} \mathbf{u} \cdot \mathbf{J} = \frac{1}{4\pi c} \mathbf{u} \cdot \nabla \times \mathbf{B}$$

On account of the fact that both the electric field \mathbf{E}' and the charge density δ' vanish in the comving fame, they must, according to the Lorentz Transformations, have nonzero values in the laboratory frame. This limit is also known as ideal *Magnetohydrodynamics* or MHD, where

$$|\mathbf{B}| \approx \frac{cL\delta}{U} \approx \frac{c}{U} |\mathbf{E}| \gg |\mathbf{E}|$$

It of course is the limiting case that is found most often in astrophysics. The retention of finite σ terms in either limit provides a resistive, or dissipative, extension to both EHD and MHD.

The determination of the scalar conductivity, σ , or more complicated forms of the generalized Ohm's Law lives in the realm of plasma physics and kinetic theory. As our philosophy in these notes has been to focus on the macroscopic, it will suffice here to simply provide an all purpose formula that will suffice for ionized gases dominated by Hydrogen,

$$\sigma \approx 10^7 \left(\frac{T}{1 \text{ deg K}}\right)^{3/2} \text{ sec}^{-1} ,$$

or in terms of the magnetic diffusivity

$$\eta \equiv \frac{c^2}{4\pi\sigma} \approx 5 \times 10^{12} \left(\frac{T}{1 \text{ deg K}}\right)^{-3/2} \text{ cm}^2 \text{ sec}^{-1}$$
.

These formulae provide ample justification for the dominance of the MHD (as opposed to EHD) limit in astrophysical settings and the general neglect of the resistive dissipation of magnetic fields.

3. Radiation: LTE, Planck, Kramers and Their Friends

The photon's frequency ν and direction of propagation **n** combine to form a 4-vector for the momentum of a photon. Thus the Doppler Shift in frequency between frames is

$$\nu' = \nu \left(1 - \frac{\mathbf{n} \cdot \mathbf{u}}{c}\right) + \cdots,$$

and the aberration is

$$\mathbf{n}' = \frac{\nu}{\nu'} \left(\mathbf{n} - \frac{\mathbf{u}}{c} \right) = \left[\mathbf{n} \left(1 + \frac{\mathbf{n} \cdot \mathbf{u}}{c} \right) - \frac{\mathbf{u}}{c} \right] + \cdots ,$$

correct to order u/c. The additional term (containing two factors of **n**) in the last expression is on account of the fact that the product ν **n** transforms as the spatial part of the 4-vector, not **n** itself. Notice because **n**' is a two-dimensional

vector, and as each component of ${\bf n}'$ carries a factor of $\nu/\nu',$ the photon phase space volume element

$$\nu d\nu d\mathbf{n} = \nu' d\nu' d\mathbf{n}'$$

is an invariant under the Lorentz Transformation. A very useful fact when comparing moments.

The transformation properties between the specific intensity $I'_{\nu'}(\mathbf{x}', t'; \mathbf{n}')$, and $I_{\nu}(\mathbf{x}, t; \mathbf{n})$, and here I write out all the dependencies to impress upon you that *all* these quantities are undergoing transformations between the two frames, is simple in form but not so simple to derive

$$rac{I_{
u'}'({f x}',t';{f n}')}{
u'^3} = rac{I_
u({f x},t;{f n})}{
u^3} \; ,$$

another Lorentz invariant. This result can be derived from the premise that while observers in two different frames disagree about many things—photon frequency, propagation direction, volumes, simultaneity and so forth—they must agree on the *number* of photons present. And finally the analogous invariant associated with the opacity must be

$$\nu' \chi'_{\nu'}(\mathbf{x}', t'; \mathbf{n}') = \nu \chi_{\nu}(\mathbf{x}, t; \mathbf{n}) ,$$

in order for the transfer equation to be invariant under the Poincaré Group. From these two expressions, it follows that the emissivity must transform like

$$rac{\eta_{
u'}'({f x}',t';{f n}')}{
u'^2} = rac{\eta_
u({f x},t;{f n})}{
u^2} \; .$$

These three results are key! Suppose, in the simplest of all possible circumstances, that the opacity is isotropic (i.e., independent of \mathbf{n}') in the comoving frame, let's call it simply $\chi'_{\nu'} \equiv \chi_0(\nu')$. Then in the laboratory frame, the opacity is *not* isotropic

$$\chi_{\nu}(\mathbf{n}) = \left(1 - \frac{\mathbf{u} \cdot \mathbf{n}}{c}\right) \chi_0(\nu')$$

but contains an additional term proportional to the fluid velocity, \mathbf{u} , and it depends upon the Doppler shifted frequency. Since the Doppler shift is small in some sense, if $\chi_0(\nu')$ is a fairly smooth function (which it decidedly is *not* in the vicinity of a spectral line) then we can replace $\chi_0(\nu')$ by its Taylor series expansion about the frequency ν in the laboratory frame:

$$\chi_{\nu}(\mathbf{n}) = \left(1 - \frac{\mathbf{u} \cdot \mathbf{n}}{c}\right) \left[\chi_{0}(\nu) + (\nu' - \nu)\frac{d\chi_{0}}{d\nu} + \frac{1}{2}(\nu' - \nu)^{2}\frac{d^{2}\chi_{0}}{d\nu^{2}} + \cdots\right] .$$

Neglecting terms of order u^2/c^2 we find

$$\chi_{\nu}(\mathbf{n}) = \chi_0(\nu) - \frac{\mathbf{u} \cdot \mathbf{n}}{c} \left[\chi_0(\nu) + \nu \frac{d\chi_0}{d\nu} \right] + \cdots$$

We now have succeeded in expressing the right side of the equation in terms of photon characteristics in the laboratory frame.

In more realistic situations, for example, we need to use the Doppler Shift and aberration formulae in the functional forms for the opacity in the comoving frame to derive the correct opacity in the laboratory frame! Rest assured, this is not something you would like to simply intuit from first principles unless you are very very good at it!

A simple, but still applicable form for the isotropic emissivity in the comoving frame is

$$\eta'_{\nu'} \equiv \kappa_0(\nu') B_{\nu'}[T'] + \sigma_0(\nu') J'_{\nu'}$$

where

$$\chi'_{\nu'} \equiv \chi_0(\nu') = \kappa_0(\nu') + \sigma_0(\nu') \; .$$

The first term accounts for LTE thermal emission according to the Planck Function, while the second permits isotropic scattering. Thomson scattering of photons by free electrons is a familiar process which can be described in this fashion.

To determine the emissivity in the laboratory frame, $\eta_{\nu}(\mathbf{n})$ we proceed as we did for the opacity by first using the transformation formula

$$\eta_{\nu}(\mathbf{n}) = \left(1 + 2\frac{\mathbf{u} \cdot \mathbf{n}}{c}\right) \left[\kappa_0(\nu') B_{\nu'}[T'] + \sigma_0(\nu') J_{\nu'}'\right] ,$$

and then expressing the quantities in square brackets in terms of the photon characteristics in the laboratory frame. The thermal emission term can again be Taylor series expanded about ν . We again retain terms of order u/c, and are helped by the fact that the discrepency between the temperatures in the laboratory frame and the comoving frames differ to leading order by u^2/c^2 , hence we can set T' = T in the Planck Function. The scattering term is more complicated owing to the presence of the mean intensity in the comoving frame:

$$J_{\nu'}' = \frac{1}{4\pi} \oint d\mathbf{n}' \ I_{\nu'}'(\mathbf{n}') = \frac{1}{4\pi} \oint d\mathbf{n}' \left(\frac{\nu'}{\nu}\right)^3 I_{\nu}(\mathbf{n}).$$

Neglecting terms of order u^2/c^2 this simplifies to

$$I_{\nu'}' = \frac{1}{4\pi} \oint d\mathbf{n}' \left(1 - 3\frac{\mathbf{u} \cdot \mathbf{n}}{c} \right) I_{\nu}(\mathbf{n})$$

We now need to transform $d\mathbf{n}'$ to $d\mathbf{n}$. The fast and dirty way to do this is to use the invariance of the volume element

$$\nu d\nu d\mathbf{n} = \nu' d\nu' d\mathbf{n}'$$

to infer that

$$d\mathbf{n}' = \left(\frac{\nu}{\nu'}\right)^2 d\mathbf{n} \; .$$

The careful way to do this is to set up the integration over solid angles using **u** as the polar axis and defining the polar angles ϑ for **n** and ϑ' for **n'**. Then, the aberration formula

$$\mathbf{n}' = \left[\mathbf{n}\left(1 + \frac{\mathbf{n} \cdot \mathbf{u}}{c}\right) - \frac{\mathbf{u}}{c}\right] + \cdots,$$

implies

$$\cos\vartheta' = \cos\vartheta\left(1 + \frac{u}{c}\cos\vartheta\right) - \frac{u}{c}$$

Either way, one finds

$$J_{\nu'}' = \frac{1}{4\pi} \oint d\mathbf{n}' \left(1 - \frac{\mathbf{u} \cdot \mathbf{n}}{c} \right) I_{\nu}(\mathbf{n}) = J_{\nu} - \frac{1}{c} \mathbf{u} \cdot \mathbf{H}_{\nu} + \cdots$$

correct to order u/c. Using the Taylor series expansion for the factor $\sigma_0(\nu')$ we have, collecting all our results and omitting terms of order u^2/c^2

$$\eta_{\nu}(\mathbf{n}) = \kappa_{0}(\nu)B_{\nu}[T] + \sigma_{0}(\nu)J_{\nu} - \frac{1}{c}\sigma_{0}(\nu)\mathbf{u}\cdot\mathbf{H}_{\nu} + \frac{\mathbf{u}\cdot\mathbf{n}}{c}\left[3\kappa_{0}(\nu)B_{\nu}[T] + J_{\nu}\left(2\sigma_{0}(\nu) - \nu\frac{d\sigma_{0}}{d\nu}\right) - \frac{d}{d\nu}\left(\nu\kappa_{0}(\nu)B_{\nu}[T]\right)\right] + \cdots$$

which combined with

$$\chi_{\nu}(\mathbf{n}) = \kappa_0(\nu) + \sigma_0(\nu) - \frac{\mathbf{u} \cdot \mathbf{n}}{c} \left[\kappa_0(\nu) + \sigma_0(\nu) + \nu \frac{d\kappa_0}{d\nu} + \nu \frac{d\sigma_0}{d\nu} \right] + \cdots$$

completes our specification of the right side of the transfer equation in the laboratory frame. Pause and savor the result—we have journeyed a long way to get here! The velocity-dependent contributions are rich in physics and implications.

Although we have provided a basic working program for the source function, we have not said anything about the opacity up to this juncture. Like the electrical conductivity, this is a very complex microscopic calculation involving all sorts of issues like material composition, degree of ionization and so forth. Again, in keeping with our philosophy, the best approach is to provide you with some working rules of thumb that will not get you into terrible trouble. If you need to do better, then you need to look into the microphysics.

In any case, a fairly careful consideration of the transfer of radiation in astrophysics suggests that there are three distinct sources of opacity which dominate at low (3000 K $\leq T \leq 8000$ K), intermediate (2 × 10⁴ K $\leq T \leq 10^{7.5}$ K) and high ($T \leq 10^{7.5}$ K) temperature regimes. At high temperatures, things are the simplest, since most elements are fully ionized and *scattering* from free electrons dominates the opacity. At intermediate temperatures bound-free (and to a lesser extent free-free) absorption/emission of photons by atoms and ions is much more important than scattering from free electrons (which are starting to become scarcer as the temperature decreases). First ionization of most elements is still fairly complete. The formula used here is named after Kramers who was the first to attempt to derive it. And finally, at the lowest temperatures, in what must have struck people as a cruel twist of fate, bound-free absorption/emission off the negatively charged anion of Hydrogen, H⁻, is the leading source of opacity!

Ingenious fits have been derived for these quantities in terms of three compositional parameters, X, Y, and Z: the fractional abundances of Hydrogen (X), Helium (Y) and everything else (Z), which must sum to one, X + Y + Z = 1. For the solar atmosphere we can use X = 0.7381, Y = 0.2485 and Z = 0.0134.

Here they are:

$$\left(\frac{\chi}{\rho}\right)_{H^{-}} \approx 1.1 \times 10^{-25} Z^{1/2} \left(\frac{\rho}{1 \text{ gm cm}^{-3}}\right)^{1/2} \left(\frac{T}{1 \text{ deg K}}\right)^{77/10} \text{ cm}^2 \text{ gm}^{-1} ,$$

and, yes, that is 77/10,

$$\left(\frac{\chi}{\rho}\right)_{\text{Kramers}} \approx 4 \times 10^{25} (1+X) (Z+0.001) \left(\frac{\rho}{1 \text{ gm cm}^{-3}}\right) \left(\frac{T}{1 \text{ deg K}}\right)^{-7/2} \text{ cm}^2 \text{ gm}^{-1} ,$$
$$\left(\frac{\chi}{\rho}\right)_{\text{Thomson}} \approx 0.2 (1+X) \kappa_R(\rho, T) \text{ cm}^2 \text{ gm}^{-1} ,$$

where

$$\frac{1}{\kappa_R(\rho,T)} \equiv \left[1 + 2.7 \times 10^{11} \left(\frac{\rho}{1 \text{ gm cm}^{-3}}\right) \left(\frac{T}{1 \text{ deg K}}\right)^{-2}\right] \left[1 + \left(\frac{T}{4.5 \times 10^8 \text{ deg K}}\right)^{43/50}\right]$$

is a relativistic correction which accounts for the fact that the Thomson cross section goes over to the Klein-Nishina cross section when the energy of the photon starts to become comparable to the rest mass of the electron. And, by the way, the scattering is no longer isotropic in the rest frame of the electrons.

Finally for those of you who really like to work in cold environments, between 1500 K and 3000 K, molecules like H_2O and CO provide most of the opacity and the requisite formula is again very simple

$$\left(\frac{\chi}{\rho}\right)_{H_2O,CO} \approx 0.1 \ Z \ \mathrm{cm}^2 \ \mathrm{gm}^{-1} \ .$$

As you notice, there is a gap in our formulae. Between 8000 K and 2×10^4 K it is necessary to interpolate between the H⁻ and Kramers opacities. And this is particularly difficult because Kramers assumes complete ionization of the major constituents, while partial ionization applies in this temperature gap. One of the previous homework exercises in Act I Scene 4 allows you to use MURaM to carry out this interpolation. Otherwise, there is no recourse except to dig into the literature on opacity tables.

Notice that Thomson scattering can be retained all the way into the intermediate temperature regime and the gap provided we use

$$0.2(1+X) = \frac{N^e \sigma_T}{\rho}$$

where

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2}\right)^2 = 6.6525... \times 10^{-25} \text{ cm}^2$$

is the Thomson cross section. The right side of the previous equation, where N^e is the electron number density [dimensions: cm⁻³], is always true, while the right side is correct when there is full ionization of Hydrogen. Hence, as the temperature decreases toward 10⁴ K and below, the left side of this equation becomes increasingly inaccurate as the few free electrons are provided by Sodium, Potassium and Iron, and Hydrogen is actually working at cross purposes by populating the H⁻ anion with some of these electrons donated by the low ionization potential metals.

4. Transport: Know Who's Carrying the Energy vs Momentum

Here are some useful expressions for the transport coefficients for viscosity

$$\mu = 1.2 \times 10^{-16} \left(\frac{T}{1 \text{ deg K}}\right)^{5/2} \text{ gm cm}^{-1} \text{ sec}^{-1}$$
,

and thermal conduction

$$\kappa = 2 \times 10^{-6} \left(\frac{T}{1 \text{ deg K}} \right)^{5/2} \text{ gm cm sec}^{-3} \text{ deg K}^{-1}$$

They are valid for fully ionized Hydrogen, but will work reasonably well under solar atmospheric conditions. If you are working in the neutral part of the Earth's atmosphere, the following approximation is not bad,

$$\kappa = 10^2 \left[43.8 + \frac{1}{20} \left(\frac{T}{1 \text{ deg K}} \right) + 16590 \left(\frac{p}{1 \text{ bar}} \right) \left(\frac{T}{1 \text{ deg K}} \right)^{-1} \right] \text{gm cm sec}^{-3} \text{ deg K}^{-1}.$$

To compare the efficacy of thermal conduction and viscosity in transporting energy, we form the dimensionless number

$$\frac{|\mathbf{u} \cdot \boldsymbol{\sigma}|}{|\mathbf{\kappa} \cdot \nabla T|} \approx \frac{\mu u^2}{\kappa T} = 6 \times 10^{-11} \left(\frac{T}{1 \text{ deg K}}\right)^{-1} \left(\frac{u}{1 \text{ cm/sec}}\right)^2$$

which indicates that except at very high flow speeds and extremely low temperatures, viscosity is less important than thermal conduction. (For those who relish dimensionless numbers, this is essentially the product of the Prandtl Number and the square of the Mach Number.) For example, let's take $T = 10^5$ K where Hydrogen is fully ionized, then we need flow speeds $u \approx 0.01c$ in order that this ratio is close to one. The reason for this is that free electrons are very efficient at carrying energy, but it requires the less mobile protons to carry the momentum. In thermal equilibrium, the proton thermal speed is smaller than the electron thermal speed by $\sqrt{m_e/m_p} \approx \sqrt{1836} = 43$.

Conversely for an ideal gas composed of neutral Hydrogen, say, this imbalance is not so lopsided, because the same particle, the Hydrogen atom, has to transport *both* the energy and the momentum. This underscores the importance of understanding the basic microphysics of your material before you decide which terms to keep and which to discard.

5. Summary

This Scene has provided a quick and dirty approach to closing our RMHD equations by deriving some rules of thumb for estimating $\{\delta, \mathbf{J} | \chi_{\nu}, \eta_{\nu} | p, T, \sigma, \kappa\}$. This is microphysics pure and simple. It is a whole other course in fact.

And it brings down the curtain on our second Act of this Opera.

The two equations to take away from all of this are the expressions developed in §3 for what the emissivity and the opacity look like in the laboratory frame if they are supposed to be isotropic in the comoving frame. Each comes equipped with terms proportional to the fluid velocity **u**. Had we gone beyond the first order in u/c, there would be terms quadratic in **u** as well. Luckily, for my ability to IATEX these equations, we stopped where we did. In any case, practically, if one needs terms at this order, one should be treating the problem from a fully relativistic formulation to begin with.

You have now, in the immortal words of my late father, enough information to be able to hang yourself—figuratively speaking of course. Well, at any rate, you have enough information and insight on astrophysical systems where gravity, radiation and large-scale electromagnetic fields play meaningful roles, to be able to assess any problem you come across and make some sensible decisions about how to ascertain solving the questions that such systems pose. You may not actually be able to answer those questions—such systems can be very complicated as we have seen—but what you will *not* do is make the mistake of handling them incorrectly. And this, my friends, is actually quite a remarkable talent to have. You have many colleagues who do not share this talent with you. Use it well.

In the third Act, which follows, therefore, I present no libretto for you. Instead, I will share some references to the small sample of problems which I will treat in the lecture. But you can write the libretto yourself at this point, and my assistance is no longer required. Congratulations!

But, P.S., don't skip the Appendix as a critical final plot twist is revealed...

6. Exercises

None! Take a break...you earned it.

7. Further Reading

The classic reference on transport coefficients for astrophysical plasmas is [S 6] Lyman Spitzer, Jr., Physics of Fully Ionized Gases, 2nd Edn, (New York, NY: John Wiley & Sons; 1962), xii+176,

which is dated, out-dated and has been superseded on so very many fronts. But all of these advances post-Spitzer require that one invest significant time and effort in understanding the microsphysics. Parker's <u>Conversations</u> [**P** 3] comes closest in spirit to Spitzer's approach of getting the most out of a minimal investment in machinery and formalism. Both Shu [**S** 1] and Mihalas & Mihalas [**MM** 1] also try to stay out of the weeds on this topic. For machinery, formalism, and weeds, Gombosi [**G** 1] and Vincenti & Kruger [**VK** 1] are good places to start.

Speaking of weeds, the computation of opacities and effective equations of

state (which you will notice we quietly avoided in this Scene) for different admixtures of elements is an entire industry on its own. Try, for example,

[**BL 1**] P. Burger & H.J.G.L.M. Lamers, "Analytical espressions for the Rosselandmean opacity and electron scattering in stellar atmospheres", *Astronomy & Astrophysics*, **218**, 161-6. 1989.

If you look through the Collected Bibliography, several of the entries will stand out as places where you can begin to get a sense of the scope of this industry. Because I have not mentioned them so far, and they are deserving of mention to be sure, the two volumes

[CG 1] John P. Cox (with R. Thomas Giuli), Principles of Stellar Structure. Volume I: Physical Principles. Volume II: Applications to Stars., (New York, NY: Gordon and Breach Science Publishers; 1968), xxliv+1327,

are an essential component of any astrophysicist's library. Their importance can be appreciated from the following anecdote. Professor David B. Guenther, now of St. Mary's University in Halifax, Nova Scotia, and I were both postdocs together in the early 1980's in Boulder. David spied the two volumes of Cox's magnum opus on my shelf and offered me in trade the entire 8 volumes (volume 4 came in two separate bindings at the time—so this is a 9:2 trade) in hardback of Landau & Lifshitz's Course of Theoretical Physics. Speaking of which, I have not mentioned this series, along with the 9th volume entitled Physical Kinetics, where Landau goes missing in action. They are amazing of course and you should own them all and read them from cover to cover. My professors at the University of Chicago were fond of saying that not a single word in these volumes (this, being before the 9th appeared) can be attributed to Landau, and not a single idea can be credited to Lifshitz. Perhaps they were being too harsh. Incidentally, there is a *four* volume set that covers the same material by another Russian named Benjamin G. Levich, a student of Lev Landau's, entitled Theoretical Physics published in 1971 by Elsevier, which is in some sense even nicer than the full gamut of Landau & Litshitz.

The so-called calculus of variations is a beautiful subject that has many applications. As physics students we encounter it first in mechanics, when the concepts of Lagrangeans, Hamiltonians and principles of least action emerge from Newton's quasi-engineering approach of driving accelerations with forces. And, of course, we carry these concepts over into quauntum mechanics when we wite $H\Psi = E\Psi$. From a purely mathematical viewpoint, however, the calculus of variations has to do with finding the extrema of *functionals*, which are mappings from vector spaces of functions, usually a Hilbert Space meaning that there is an inner product defined on the function space, to a scalar field, like \mathbb{R} . For this mathematical approach, try Chapter IV of

[CH 1] R. Courant & D. Hilbert, Methods of Mathematical Physics. Volume I. (New York, NY: Interscience Publishers; 1953), xv+561,

and

[A 1] A.M. Arthurs, Complementary Variational Principles, (Oxford, UK: Clarendon Press; 1970), ix+95.

A book that is extremely rich in content, and which ties together so many

of the different concepts we have introduced in this Opera is

[A 2] V.I Arnold, <u>Mathematical Methods of Classical Mechanics</u>, (New York, NY: Springer-Verlag; 1984), x+462.

Finally, variational methods are discussed by [**DK 1**], [**GKP 1**], [**GS 1**], [**K 1**], [**KD 1**], [**O 1**], [**P 8**], [**R 1**], [**S 1**] and [**W 1**].

If you are still reading these notes at this point, you have earned a treat. J.B.—or John Bradbury—Sykes was a man of many talents: astrophysicist, crossword puzzle champion, linguist and translator. He painstakingly translated the entire Landau & Lifshitz series, into English, for example. And if you have an eye for detail, you will notice that he helped Professor Davison with his book on neutron transfer. J.B. also had an exceedingly fine sense of humor, which led to the following hilarious parody of a typical Chandrasekhar paper:

[C 5] S. Candlestickmaker, "On the imperturbability of elevator operators. LVII", *Quarterly Journal of the Royal Astronomical Society*, **13**, 63-6, 1972.

8. Appendix A: Variational Principles & Energy Principles

This topic needed to go somewhere in the first two Acts of this *Opera* and somehow it ended up here. It is not particularly essential to anything that comes before or after, but it is something that one needs to be familiar with at a minimum. So, consider this the minimum.

Consider a Hilbert space, \mathcal{H} , whose elements are functions $\mathbf{y} : [a, b] \to \mathbb{R}$, where $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b, a, b \in \mathbb{R}\}$. A functional $J[\mathbf{y}]$ on \mathcal{H} is a mapping $J : \mathcal{H} \to \mathbb{R}$. Of interest here, are functionals of the form

$$J[\mathbf{y}] \equiv \int_{a}^{b} dx \ \mathfrak{L}(\mathbf{y}, \mathbf{y}', x) \ ,$$

where $\mathbf{y}' \equiv d\mathbf{y}/dx$. So the functions in our Hilbert Space must be (twice) differentiable (at a minimum). We call \mathfrak{L} the *density* associated with the functional J, and require that it, too, be differentiable with respect to its three arguments. A concrete example would be

$$\mathfrak{L} = \frac{1}{2} (\mathbf{y'}^2 + \mathbf{y}^2) \; ,$$

which is differentiable any number of times.

For a given $\{J, \mathfrak{L}\}$ we now seek those elements $\mathbf{y} \in \mathcal{H}$ which render J stationary with respect to small variations around \mathbf{y} , subject to the condition that $\mathbf{y}(a) = \alpha$ and $\mathbf{y}(b) = \beta$. Let \mathbf{z} be such an element, and $\boldsymbol{\xi}$ some other element in \mathcal{H} . If we put $\mathbf{y} = \mathbf{z} + \boldsymbol{\xi}$ in the above expression for \mathfrak{L} we have

$$\mathfrak{L} = \frac{1}{2} ({\mathbf{z}'}^2 + {\mathbf{z}}^2) + {\mathbf{z}'} {\boldsymbol{\xi}'} + {\mathbf{z}} {\boldsymbol{\xi}} + \frac{1}{2} ({{\boldsymbol{\xi}'}}^2 + {\boldsymbol{\xi}}^2) \ .$$

Substituting this expression in the definition of the functional, we have

$$J[\mathbf{z}] - J[\boldsymbol{\xi}] = \int_a^b dx \ (\mathbf{z}'\boldsymbol{\xi}' + \mathbf{z}\boldsymbol{\xi}) + \mathbf{z}'(b)\boldsymbol{\xi}(b) - \mathbf{z}'(a)\boldsymbol{\xi}(a) - \int_a^b dx \ (\mathbf{z}'' - \mathbf{z})\boldsymbol{\xi} \ .$$

Since $J[\boldsymbol{\xi}] = O(\boldsymbol{\xi}^2)$, the right side of this equation must vanish for J to be stationary for deviations from the desired solution \mathbf{z} . Because $\mathbf{y}(a) = \mathbf{z}(a) = \alpha$ and $\mathbf{y}(b) = \mathbf{z}(b) = \beta$, it follows that $\boldsymbol{\xi}(a) = \boldsymbol{\xi}(b) = 0$.

Since $\pmb{\xi}$ is otherwise arbitrary, we require

$$\left(\frac{d^2}{dx^2} - 1\right)\mathbf{z} = 0$$

subject to the boundary conditions

$$\mathbf{z}(b) = \beta, \qquad \mathbf{z}(a) = \alpha.$$

The solution is

$$\mathbf{z} = \beta \frac{\sinh(x-a)}{\sinh(b-a)} + \alpha \frac{\sinh(b-x)}{\sinh(b-a)} \,,$$

and because $J[\boldsymbol{\xi}] \ge 0$ for arbitrary $\boldsymbol{\xi}$, this **z** in fact *minimizes* the functional.

For an *arbitrary*, $\mathfrak{L}(\mathbf{y}', \mathbf{y}, x)$, we have

$$J[\mathbf{z} + \boldsymbol{\xi}] = \int_{a}^{b} dx \, \boldsymbol{\xi} \left(\frac{\partial \mathfrak{L}}{\partial \mathbf{z}} - \frac{d}{dx} \frac{\partial \mathfrak{L}}{\partial \mathbf{z}'} \right) \, + \\ \frac{1}{2!} \int_{a}^{b} dx \, \left(\boldsymbol{\xi}'^{2} \frac{\partial^{2} \mathfrak{L}}{\partial \mathbf{z}'^{2}} + 2\boldsymbol{\xi}' \boldsymbol{\xi} \frac{\partial^{2} \mathfrak{L}}{\partial \mathbf{z} \partial \mathbf{z}'} + \boldsymbol{\xi}^{2} \frac{\partial^{2} \mathfrak{L}}{\partial \mathbf{z}^{2}} \right) + \\ \frac{1}{3!} \int_{a}^{b} dx \, \left(\boldsymbol{\xi}'^{3} \frac{\partial^{3} \mathfrak{L}}{\partial \mathbf{z}'^{3}} + 3\boldsymbol{\xi}'^{2} \boldsymbol{\xi} \frac{\partial^{3} \mathfrak{L}}{\partial \mathbf{z} \partial \mathbf{z}'^{2}} + 3\boldsymbol{\xi}' \boldsymbol{\xi}^{2} \frac{\partial^{3} \mathfrak{L}}{\partial \mathbf{z}^{2} \partial \mathbf{z}'} + \boldsymbol{\xi}^{3} \frac{\partial^{3} \mathfrak{L}}{\partial \mathbf{z}^{3}} \right) + \cdots$$

where the structure of the remaining terms in the " \cdots " is clear. For our concrete example above, the third and all higher terms vanish. J is stationary if the first term is zero, which yields the so-called *Euler-Lagrange Equation*:

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{z}} = \frac{d}{dx} \frac{\partial \mathfrak{L}}{\partial \mathbf{z}'} \; ,$$

a second-order ordinary differential equation for $\mathbf{z},$ subject to the boundary conditions

$$\mathbf{z}(b) = \beta, \qquad \mathbf{z}(a) = \alpha.$$

The sign of the second term indicates whether this stationary point is an extrema (if it is positive or negative definite for all $\boldsymbol{\xi}$) or a point of inflection. There is a very beautiful theorem which provides a necessary requirement for the second term to be positive definite, called *Legendre's Condition*. And the third term can be used to study wave-wave coupling problems!

This is the simplest of all variational problems. We can generalize this discussion in three different directions (simultaneously in fact if desired) to create a richer and more complicated theory. *First.* we can increase the order of the derivatives of \mathbf{y} which appear in the density

$$\mathfrak{L}(\mathbf{y}^{(n)},\mathbf{y}^{(n-1)},\cdots\mathbf{y}'',\mathbf{y},x)$$
 .

Second, we can increase the number of independent variables, $x \to \{x_1, x_2, \dots, x_m\}$ and $\mathbf{y}' \to \{\mathbf{y}'_1, \mathbf{y}'_2, \dots, \mathbf{y}'_m\}$, where $\mathbf{y}'_m \equiv \partial \mathbf{y} / \partial x_m$, which appear in the density,

$$\mathfrak{L}(\mathbf{y}'_1, \mathbf{y}'_2, \cdots, \mathbf{y}'_m, \mathbf{y}, x_1, x_2, \cdots, x_m) \ .$$

Third we can increase the number of dependent variables that appear in the density $\mathbf{y} \rightarrow \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_i\}$

$$\mathfrak{L}(\dot{\mathbf{y}}_1, \dot{\mathbf{y}}_2, \cdots \dot{\mathbf{y}}_i, \mathbf{y}_1, \mathbf{y}_2, \cdots \mathbf{y}_i, x)$$
.

where $\dot{\mathbf{y}}_i \equiv d\mathbf{y}_i/dx$. The replacement " $\mathbf{y} = \mathbf{z} + \boldsymbol{\xi}$ " now yields a very complicated sum for J and a rich variety (and number!) of Euler-Lagrange Equations.

The variational approach to RMHD employs a combination of the last two of the three generalizations. In place of the single independent variable x above, we use the full 3+1 (that is *four*) dimensional space-time, so $dx \rightarrow d\mathbf{x}dt$. In place of the single dependent variable \mathbf{y} we use the *three* components of the Lagrangean mapping $X_i(\mathbf{x}', t)$, that assigns a fluid element originally at position \mathbf{x}' at time t = 0 to a position $\mathbf{x} = \mathbf{X}(\mathbf{x}', t)$ at some later time $t \ge 0$. So, our density may contain up to 4 independent variables, 3 dependent variables, 4×3 first partial derivatives of the dependent with respect to the independent variables, for a grand total of 19 arguments.

With this in mind, it then comes as a major surprise (to me at least) to discover than RMHD can be obtained by finding the extrema of the functional

$$J_{\rm RMHD}[\mathbf{X}] = \int d\mathbf{x} dt \left\{ \frac{1}{2} \rho |\mathbf{u}|^2 - \rho(e+\Phi) - \frac{1}{8\pi} (|\mathbf{E}|^2 + |\mathbf{B}|^2) - \frac{c}{4\pi} J \right\} ,$$

over some appropriate subset of space-time. The governing equations for $\mathbf{X}(\mathbf{x}', t)$ follow from the analogous Euler-Lagrange Equation. More amazing yet, is that the *stability* of an extema, say $\mathbf{X} = \mathbf{Z}$, of this functional can be determined by analyzing the *second* term in the analogous expansion of $J_{\text{RMHD}}[\mathbf{Z} + \boldsymbol{\xi}]$. Such an analysis is sometimes called the *energy principle* particularly for MHD. And, better yet, the *third* term in the analogous expansion can be used to determine mode-mode coupling between stable perturbations (i.e., waves) that live on the extrema solution \mathbf{Z} !

All really good *Opera*'s often have that "*aha!*" moment near the end of the second Act. Ours is no exception. The "*aha!*" moment for us is the realization that the expression on the right of the equation

$$\mathfrak{H}_{\text{RMHD}} = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho(e+\Phi) + \frac{1}{8\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2) + \frac{c}{4\pi}J$$

is immediately recognized as the *total* energy density of the material, gravitational, electromagnetic, and radiation fields! The quantity on the left of this equation is then the *Hamiltonian Density* corresponding to the *Lagrangean Density*,

$$\mathfrak{L}_{\text{RMHD}} \equiv \frac{1}{2}\rho |\mathbf{u}|^2 - \rho(e+\Phi) - \frac{1}{8\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - \frac{c}{4\pi}J \;.$$

Therefore, the internal material energy, gravitational, electromagnetic and radiation fields can be thought of as storing *potential energy* that can be turned into *kinetic energy* of the material and vice-versa! Pause and reflect upon this poignant moment as the curtain drops on Act II.

While all of this is true, the apparent simplicity of this amazing sequence of plot revelations is unfortunately illusory. To express any particular term that appears in the RMHD Lagrangean Density

$$\mathfrak{L}_{\text{RMHD}} \equiv \frac{1}{2}\rho|\mathbf{u}|^2 - \rho(e+\Phi) - \frac{1}{8\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - \frac{c}{4\pi}J$$

in terms of **X** and all of its partial derivatives is not a trivial undertaking. Remember, from Act I Scene 1, that the density varies as the inverse of the determinant of the Jacobian J of the mapping, $\partial X_i / \partial x'_j$,

$$\rho(\mathbf{x}',t) = \frac{\rho_0(\mathbf{x}')}{J(\mathbf{x}',t)} = \frac{\rho(\mathbf{x}',0)}{J(\mathbf{x}',t)}$$

and if

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

then

$$B_k(\mathbf{x}', t) = \frac{B_j(\mathbf{x}', 0)}{J(\mathbf{x}', t)} \frac{\partial X_k}{\partial x'_j},$$

where $\mathbf{B}_0(\mathbf{x}') = \mathbf{B}(\mathbf{x}', t = 0)$ is the initial magnetic field. While

$$\mathbf{u}(\mathbf{x}',t) = \frac{\partial \mathbf{X}}{\partial t} \; ,$$

the derivation of the required expressions for the internal energy per unit mass, the gravitational potential, and the mean intensity are much more elaborate. Notice too that all of the expressions we have derived so far depend upon the initial set of coordinates \mathbf{x}' , suggesting that we may wish to transform the integration domain from the initial to the current position of a fluid element,

$$d\mathbf{x} = \frac{1}{|J(\mathbf{x}', t)|} d\mathbf{x}' \; .$$

To pursue these matters further, consult Ogilvie [O 1], Keppens & Demaerel [KD 1] and Webb [W 1], for example.