Radiation Magnetohydrodynamics: Synopsis

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The **mass** density is

$$\rho \ ,$$

and the mass flux vector is

$$\rho \mathbf{u} \ .$$

The **momentum** density vector is

$$\rho \mathbf{u} + \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} + \frac{1}{c^2} \int_0^\infty d\nu \oint d\mathbf{n} \ \mathbf{n} I_\nu \ ,$$

and the momentum flux tensor is

$$p \mathbb{1} + \rho \mathbf{u}\mathbf{u} - \mathbb{\sigma} + \mathbb{G} + \mathbb{M} + \mathbb{P} \ ,$$

where

$$\mathbb{\sigma} = \mu \left(\nabla \mathbf{u} + [\nabla \mathbf{u}]^T \right) - \frac{2}{3} \mu \nabla \cdot \mathbf{u} \ ,$$

$$\mathbb{G} = \frac{1}{8\pi G} \left(2 \nabla \Phi \nabla \Phi - \mathbb{1} |\nabla \Phi|^2 \right) \ ,$$

$$\mathbb{M} = \frac{1}{8\pi} \left(\mathbb{1} [|\mathbf{E}|^2 + |\mathbf{B}|^2] - 2\mathbf{E}\mathbf{E} - 2\mathbf{B}\mathbf{B} \right) \ ,$$

and

$$\mathbb{P} = \frac{1}{c} \int_0^\infty d\nu \oint d\mathbf{n} \ \mathbf{n}\mathbf{n} I_\nu \ .$$

The **energy** density is

$$\frac{1}{2} \rho |\mathbf{u}|^2 + \rho e + \frac{2 - \gamma}{2} \rho \Phi + \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} + \frac{1}{c} \int_0^\infty d\nu \oint d\mathbf{n} \ I_\nu \ ,$$

and the energy flux vector is

$$\mathbf{u} \cdot \left(\rho \frac{1}{2} |\mathbf{u}|^2 + p + \rho e + \rho \Phi - \mathbb{\sigma} \right) + \frac{\gamma}{8\pi G} \left(\frac{\partial \Phi}{\partial t} \nabla \Phi + \Phi \nabla \frac{\partial \Phi}{\partial t} \right) +$$

$$\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} - \mathbb{\kappa} \cdot \nabla T + \int_0^\infty d\nu \oint d\mathbf{n} \ \mathbf{n} I_\nu \ ,$$

where $\gamma = 0$ if the gravitational potential is provided by some external agent, and $\gamma = 1$ if self-gravity is accounted for explicitly by the Newton-Poisson Equation.

The force density that appears on the the right side of the Navier-Stokes Equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla\cdot(p - \mathbb{\sigma}) = \mathbf{f}\ ,$$

is

$$\mathbf{f} = -\rho\nabla\Phi + \delta\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B} - \frac{1}{c}\int_0^\infty d\nu \oint d\mathbf{n}\ \mathbf{n}[\eta_\nu - \chi_\nu I_\nu]\ ,$$

and the heat-source/entropy-production term that appears on the right side of the internal energy equation

$$\rho\left(\frac{\partial e}{\partial t} + \mathbf{u}\cdot\nabla e\right) + p\nabla\cdot\mathbf{u} = \rho\frac{\delta q}{\delta t} = \rho T\frac{ds}{dt}\ ,$$

is

$$\rho\frac{\delta q}{\delta t} = \nabla\cdot\mathbb{\kappa}\cdot\nabla T + \mathbb{\sigma} : \nabla\mathbf{u} + \mathbf{J}\cdot\mathbf{E} - \mathbf{u}\cdot\left(\delta\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}\right) -$$

$$\int_0^\infty d\nu \oint d\mathbf{n}\ \left(1 - \frac{\mathbf{u}\cdot\mathbf{n}}{c}\right)[\eta_\nu - \chi_\nu I_\nu]\ .$$

The auxiliary field equations are

$$\nabla^2\Phi = 4\pi G\gamma\rho\ ,$$

$$\nabla\cdot\mathbf{E} = 4\pi\delta\ , \qquad c\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\ ,$$

$$\nabla\cdot\mathbf{B} = 0\ , \qquad c\nabla\times\mathbf{B} = 4\pi\mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}\ ,$$

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = \eta_\nu - \chi_\nu I_\nu\ ,$$

and the equation of state for the matter is

$$f(p,\rho,T,e,s) = 0\ .$$

Finally, closure relations are required to determine μ, $\mathbb{\kappa}$, $\mathbf{J}$, η_ν and χ_ν. And the moral of our story, of course, is that these closure relations should *always* be reckoned in the comoving frame of the material and then subsequently transformed back to the laboratory frame by means of the appropriate Lorentz Transformations from the 10-parameter Poincaré Group which describes the Minkowski 3+1 space-time.

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