ACT I. SCENE 4: THE RADIATION FIELD

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1. Introduction

Electromagnetic radiation spans a truly remarkable range of frequencies, wavelengths and energies:

Band	$[\nu]$ Hz	$[\lambda]$ cm	$[\varepsilon] eV$	$[\varepsilon]$ erg	[T]deg K
radio	$10^{6.5}$	10^{4}	$10^{-7.9}$	$10^{-19.7}$	$10^{-3.8}$
μ – waves	$10^{9.5}$	10	$10^{-4.9}$	$10^{-16.7}$	$10^{-0.8}$
infrared	$10^{13.5}$	10^{-3}	$10^{-0.9}$	$10^{-12.7}$	$10^{3.2}$
visible	$10^{14.8}$	$10^{-4.3}$	$10^{0.4}$	$10^{-11.4}$	$10^{4.5}$
$\mathbf{x} - \mathbf{ray}$	10^{18}	$10^{-7.5}$	$10^{3.6}$	$10^{-8.2}$	$10^{7.7}$
$\gamma - ray$	$10^{20.5}$	10^{-10}	$10^{6.1}$	$10^{-5.7}$	$10^{10.2}$

all of which can, and have, been profitably treated via the statistical methods of radiative transfer.

2. The Transfer Equation

The equation of radiative transfer for the specific intensity $I_{\nu}(\mathbf{x}, t; \mathbf{n})$ [dimensions: erg cm⁻² sec⁻¹ Hz⁻¹ ster⁻¹] is

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mathbf{n} \cdot \nabla I_{\nu} = \eta_{\nu} - \chi_{\nu} I_{\nu} \ .$$

Here, $\nu \equiv \omega/2\pi$ [dimensions: Hz] is the frequency, and $\mathbf{n} \equiv \mathbf{k}/|\mathbf{k}|$ [dimensions: ster] is the unit vector parallel to the wavevector, of the photon, and c is the speed of light. The left side of this equation is the continuity equation for photons—if the right side is absent, then the number of photons is conserved. The interaction of photons with the material is accounted for by the right side of this equation. The second terms quantifies the destruction of photons. The quantity, $\chi_{\nu}(\mathbf{x}, t; \mathbf{n})$ [dimensions: cm^{-1}] is the *opacity* and is assumed to be non-negative. The creation of photons is described by the *emissivity* $\eta_{\nu}(\mathbf{x}, t; \mathbf{n})$ [dimensions: $\mathrm{erg} \ \mathrm{cm}^{-3} \ \mathrm{sec}^{-1} \ \mathrm{Hz}^{-1} \ \mathrm{ster}^{-1}$], which will also be non-negative in what follows. Both the emissivity and the opacity must be determined from some knowledge, or specification, of the microphysics of how photons intereact with the material.

Formally, one can always define the *source function*, $S_{\nu}(\mathbf{x}, t; \mathbf{n})$, by dividing the emissivity by the opacity

$$S_{\nu} \equiv \frac{\eta_{\nu}}{\chi_{\nu}}$$

which has the same dimensions as I_{ν} . Sometimes it is preferable to work with S_{ν} instead of η_{ν} , but the outcome is the same no matter which choice is made.

When the radiation does not interact with the matter, we say the medium is (very) optically-thin, and we are left with

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mathbf{n} \cdot \nabla I_{\nu} = 0 ,$$

a first-order linear PDE for one dependent variable I_{ν} and seven independent variables \mathbf{x} , t, ν , and \mathbf{n} . The methods presented in the Appendix E of Scene 1 indicate that *any* function of $\mathbf{x} - ct\mathbf{n}$, ν and \mathbf{n} (the six integrals of the motion) is a solution of this equation!

This is easy. Complications abound, however, when the right side of this equation is not zero.

3. Momentum and Energy Equations

The transfer equation is the continuity equation for photons! Its moments, taken over the frequency and wavenumber of the photons, are conveniently the momentum and energy density carried by the radiation field! To appreciate why this is the case, notice that I_{ν} is the energy per unit time that crosses a planar surface of area $d\mathbf{S}$ carried by photons with wavenumbers in a solid angle $d\mathbf{n}$ steradians, with frequencies in an interval $d\nu$ Hz. Each photon individually makes a contribution $h\nu c$ [dimensions: erg cm sec⁻¹], where h is Planck's constant, to this energy flux, and therefore the number density of photons is

$$\frac{1}{h\nu c}I_{\nu}(\mathbf{x},t;\mathbf{n}) \ .$$

Obviously the combination of (\mathbf{n}, ν) is basically equivalent to $\mathbf{p} = \hbar \mathbf{k}$ which is the momentum of a photon.

The energy density in the radiation field (per unit frequency and solid angle) is just I_{ν}/c . Therefore if we simply integrate the transfer equation over all solid angles $d\mathbf{n}$ and all frequencies $d\nu$, we immediately arrive at

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = \int_0^\infty d\nu \oint d\mathbf{n} \, \left[\eta_\nu - \chi_\nu I_\nu \right] \,,$$

where

$$E(\mathbf{x},t) \equiv \frac{1}{c} \int_0^\infty d\nu \oint d\mathbf{n} \ I_\nu \ ,$$

and

$$\mathbf{F}(\mathbf{x},t) \equiv \int_0^\infty d\nu \oint d\mathbf{n} \ \mathbf{n} \ I_\nu \ ,$$

are the energy density and the energy flux carried by the radiation field—a very nice and compact result! The right side of this equation, which is unknown until the microphysics is brought into the formulation, represents the exchange of energy between the radiation and the material.

The same arguments used above indicate that the momentum density carried by the radiation field is simply \mathbf{F}/c^2 , therefore multiplying the transfer equation by \mathbf{n}/c and integrating gives the conservation of momentum

$$\frac{1}{c^2}\frac{\partial \mathbf{F}}{\partial t} + \nabla \cdot \mathbb{P} = \frac{1}{c} \int_0^\infty d\nu \oint d\mathbf{n} \, \mathbf{n} [\eta_\nu - \chi_\nu I_\nu] \; ,$$

where

$$\mathbb{P}(\mathbf{x},t) \equiv \frac{1}{c} \int_0^\infty d\nu \oint d\mathbf{n} \ \mathbf{nn} \ I_\nu \ ,$$

is the (symmetric) radiation pressure tensor. Again, the right side of this equation which expresses the transfer of momentum between the material and the radiation field, cannot be determined without specifying the microphysics.

4. Solving the Transfer Equation. Part 1

In any event, when we do get around to determining the opacity and the source function by some means, it will be necessary to solve the transfer equation

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mathbf{n} \cdot \nabla I_{\nu} = \chi_{\nu}[S_{\nu} - I_{\nu}] \; .$$

This turns out to be, in practice, an incredibly difficult problem because, in general S_{ν} itself will depend upon I_{ν} in an extremely complicated fashion. So to gain some facility with the properties of the transfer equation, in this section we will analyze

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mathbf{n}\cdot\nabla I_{\nu} + \chi_{\nu}I_{\nu} = \chi_{\nu}S_{\nu} ,$$

with some reasonably representative choices for χ_{ν} and S_{ν} .

This, of course, is another quintessential example of a first-order PDE in (count 'em) seven independent variables and one dependent variable for a grand total of eight x_i in the parlance of the Appendix E of Scene 1. However, since ν and **n** appear merely as parameters, we know they are integrals of the motion. And **n** lives on the unit sphere, so it is two-dimensional, therefore we can reduce our problem to four x_i . A serious improvment in outlook!

Still, before jumping into Pfaffians, it is useful to make a few general observations about this equation. Consider the case where the source function vanishes everywhere. First, it is linear and homogeneous in I (we are going to drop the ν since it is a conserved parameter for the remainder of this section), so if I starts out non-negative, it can never become negative. Second, absent a source term, the material simply destroys photons and so no matter how we start things out, the end state is that $I \equiv 0$ and all the radiation is gone—eaten by the material or escaped to infinity. *Third*, we always need to keep the middle term in this equation, but the ratio of the first to the third terms can be, depending on circumstances, very very large or incredibly small. The inverse of χ can be regarded as the mean-free-path of a photon before it interacts with, and in this case, gets destroyed by, the material. For the first term to be comparable in size to the third, the radiation field must exhibit sensible temporal variations on the time it takes a photon to traverse one mean-free-path travelling at c. Because in practice c is so large, and 1/c so small, this equation is singular. A photon traverses a typical mean-free-path of 10 km at the solar photosphere, say, in 30 μ sec, and so for the first term to figure in our analysis, we would need to be concerned with variations in the radiation field on this (or shorter) time scales. Which we aren't. Therefore it is traditional to omit this first term entirely in most treatments.

Another way to appreciate this same point is to think about releasing (via the source term S) a large number of photons from some point \mathbf{x} in the solar photosphere at an instant of time. The first term in this equation will turn on immediately and become important for the next few hundred μ sec and things will then settle into a quasi steady state and this term will become negligible again. So we need to retain this term if we wish to track how the radiation field equilibrates to (rapid!) forcing by the material.

On the other hand, it could be a serious mistake to go back to the equation of energy (and to a lesser degree momentum) conservation and rub out the time derivative contribution based on this argument. The reason is that the energy density in the radiation field can become comparable to the energy density in the material at high temperatures present in O and B stars, say, and rubbing out this term will create errors in energy conservation. Photons have very little momentum per unit energy, so it is almost always safe to omit the time derivative of **F** unless one is dealing with extreme astrophysical situations.

Given $\chi(\mathbf{x}, t; \mathbf{n})$ and $S(\mathbf{x}, t; \mathbf{n})$ everywhere, our goal is to determine $I(\mathbf{x}, t; \mathbf{n})$ from

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{n} \cdot \nabla I + \chi I = \chi S ,$$

one **n** at a time (and, also, one ν at a time). We can do this because different **n**'s (like different ν 's) are not coupled (e.g., they are integrals of the motion) when $S(\mathbf{x}, t; \mathbf{n})$ is provided to us ab initio.

To be precise, our problem is to pick an \mathbf{n} and a point \mathbf{y} , say, where we want to determine $I(\mathbf{y}, t; \mathbf{n})$, for all t, based on complete knowledge of χ and S. Before doing the math, perhaps it is worth taking a moment simply to sort out what we have to end up with. For the sake of argument, let's take t=0. All the photons that pass through our point y traveling in the direction **n** have to be created on the spot by S or have just arrived from behind us coming from the $-\mathbf{n}$ direction. Of this second part of the photon population, we can go back and figure out precisely where they had to be at any earlier time $t = -t_0$, because they all travel at the same speed c. Of course, the photons that get to us are the "lucky" ones who have not been subsequently absorbed over the distance ct_0 which they had to travel. And, along the way, they have picked up companions due to contributions from S which are aligned and timed appropriately. Therefore the number of photons of this second type we have depends upon the behavior of the opacity and the source function at previous times and locations in the negative $-\mathbf{n}$ direction consistent with the fact that they all travel at the constant velocity, c. Stated another way, when we go back and integrate behind us along the line of sight, we must use the *retarded* time appropriate for each location when we reckon the opacity and the source function! This, of course, is precisely what we found for the solutions of the electromagnetic wave equations, from which we built photons, so it is satisfying that the same physics is imbedded in the equation of radiative transfer! A very nice, and in retrospect, perhaps completely obvious, result! Ok, now let's do the math.

There are three more integral of the motions, related to the fact that photons

travel at the speed of light in the direction **n**. So, to exploit this symmetry, we replace the *four* the independent variables \mathbf{x} and t by the expressions

$$\mathbf{x} = \mathbf{y} + s\mathbf{n} + \mathbf{x}$$
$$t = \frac{1}{c}(\eta + s)$$

in terms of *two* new independent variables s and η , (both are real numbers) and the two components of **x** that are orthogonal to **n**. The inverse transformations are obviously

$$s = \mathbf{n} \cdot (\mathbf{x} - \mathbf{y}) ,$$

$$\eta = ct - \mathbf{n} \cdot (\mathbf{x} - \mathbf{y}) .$$

Using the chain rule to convert the (\mathbf{x}, t) derivatives to the $(\mathbf{x}_{\perp}, s, \eta)$ coordinates we obtain

$$\frac{\partial I}{\partial s} + \chi I = \chi S$$

the single derivative being taken at fixed η and \mathbf{x}_{\perp} , which, are the three additional integrals of the motion!

Physically, we have taken advantage of the fact that the intensity of radiation at point \mathbf{y} in the \mathbf{n} direction can only depend upon photons approaching (from the negative s direction) along the (one-dimensional) line described by

$$\mathbf{x} = \mathbf{y} + s\mathbf{n} \; ,$$

and so we used the mathematics to eliminate the extraneous parts of the transfer equation. *Mathematically*, we have identified a particular (one-dimensional) manifold of our four-dimensional Galilean space-time where all the action takes place.

This equation has essentially the same form as the equation

$$\frac{dI}{ds} + \chi I = \chi S \; ,$$

which is the standard form of the transfer equation, *omitting the time derivative*, used in the standard treaments of time-independent radiative transfer! It's nice that we can get to more or less the same place retaining the full equation with no approximation. The solution of our equation

$$\frac{\partial I}{\partial s} + \chi I = \chi S \; ,$$

can proceed along similar lines, but, with some subtle and very fascinating nuances, because, after all $d/dt \neq \partial/\partial t$.

It still makes sense (following the time-independent approach) to define a dimensionless quantity called the *optical depth* by

$$\tau(s,\eta) = \int_s^0 ds' \chi(\mathbf{y} + s'\mathbf{n}, (s'+\eta)/c; \mathbf{n}) \ .$$

Distinct from the optical depth in the standard time-independent treatment, this optical depth depends upon the time (through η) and the past behavior of the opacity along the trajectory of a photon! For example, should the opacity have become very large somewhere behind the point \mathbf{y} at some time in the past, there will be a future time at the point \mathbf{y} where the intensity will drop reflecting that fact that a large number of photons were absorbed before they could get to \mathbf{y} . Physically, this equation says that we use the opacity at the *retarded time* as we integrate forward toward \mathbf{y} along the \mathbf{n} direction.

Replacing the the s-derivative (at constant η) in favor of a τ derivative gives

$$\frac{\partial I}{\partial \tau} = I - S \ ,$$

consistent with our definition that $\tau = 0$ at $\mathbf{x} = \mathbf{y}$, and τ increases as one looks farther back along the negative *s*-direction. Be careful with sign conventions here, as we could equally have picked an optical depth with a different sense.

Integrating back from $\tau = 0$ to a finite optical depth, τ_0 we get

$$I = e^{\tau - \tau_0} I_0 + \int_{\tau}^{\tau_0} d\tau' e^{\tau - \tau'} S' ,$$

where I_0 is the specific intensity at the optical depth

$$\tau_0(\eta) = \tau(s_0, \eta) = \int_{s_0}^0 ds' \chi(\mathbf{y} + s'\mathbf{n}, (s' + \eta)/c; \mathbf{n}) ,$$

where s_0 is that distance behind the point **y** where we eventually reach the back end of a cloud, or atmosphere, through which the radiation is passing. Therefore, at this point we are not only free to, but in fact, obligated to indicate what the incident radiation I_0 looks like impinging upon the cloud/atmosphere in the **n** direction back at s_0 . The solution at **y** is obtained by setting $\tau = 0$. Therefore, if $\tau_0 \gg 1$ little information about I_0 manages to survive at **y**.

We've placed a "" on the S to indicate that it too has to be evaluated at the retarded time at the appropriate optical depth for that time. This is somewhat easier than trying to explicitly write down the exact \mathbf{x} . Problem solved! As the dust settles, its worth considering a few points.

First, if nothing depends upon time then any time is as good as any other time—and there being no time like the present, so to speak, we can evaluate the opacity and the source function at the current time and forget about photon trajectories and retarded time. We then obtain the standard result quoted in all the usual textbooks. The same is true in an asymptotic sense if we turn on the opacity and the source functions everywhere at some time t_0 and maintain them at these values for all future times. The radiation field approaches the steady-state solution as the initial transients are absorbed and propagate away. The exact solution derived here allows one to estimate how long it takes to get to this steady-state depending upon where you are in the cloud/atmosphere.

Second, in the same fashion that very little of I_0 is present in $I(\tau)$ if $\tau_0 - \tau$ is in excess of even modest values like 5-10, little of the source function matters beyond $\tau' - \tau$ unless it increases as dramatically as the exponential factor

declines. When such dramatic behavior is not in play, that is when the source function varies slowly compared to the exponential of the optical depth, I depends for the most part on the local value of S and perhaps one or at most two of the derivatives of S with respect to optical depth. This is very nice when it occurs, because then the radiation field depends on local (as opposed to global) conditions. In such situations we say the radiation field is *optically-thick*. We may then employ a Taylor series expansion of the source function about the (local) point τ :

$$S(\tau',t') = S(\tau,t) + (\tau'-\tau)S^{(1)}(\tau,t) + \frac{1}{2!}(\tau'-\tau)^2S^{(2)}(\tau,t) + \cdots ,$$

where t' is the retarded time, of course. The essential point is that the $S^{(n)}$ depend upon partial derivatives of S with respect to τ and t, evaluated at the same τ and t where we are computing I and so they may be moved *outside* of the integral. At each order n, the remaining integral can be evaluated in terms of the *Incomplete Gamma Function*, as follows:

$$\int_0^{\tau_0} dw \ w^n e^{-w} \equiv n! - \Gamma(n+1,\tau_0) = n! - \int_{\tau_0}^{\infty} dw \ w^n e^{-w}$$

Most authors will simply point out that the last integral has to be small for large τ_0 and move on, but, there is some utility, I think, in being precise and seeing how the mathematics permits us to say what this really means in practice. The Incomplete Gamma Function has the power series expansion

$$\Gamma(n+1,\tau_0) = n! - \frac{\tau_0^{n+1}}{n+1} F_1(n+1;n+2;-\tau_0) ,$$

where

$$_{1}F_{1}(a;b;x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^{2}}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{x^{3}}{3!} + \cdots,$$

is called the *Confluent Hypergeometric Function*. It has many relatives where the two '1' subscripts can be replaced by other integers. And it contains other well known functions, such as

$$e^x = {}_1F_1(a;a;x) ,$$

as special cases. In any event, the series for $\Gamma(n+1,\tau_0)$ converges for all values of τ_0 but is fairly useless when, as in our case, we are contemplating largish values of τ_0 . Then, the *asymptotic* formula

$$\Gamma(n+1;\tau_0) \sim \tau_0^n e^{-\tau_0} \left(1 + \frac{n}{\tau_0} + \frac{n(n+1)}{\tau_0^2} + \cdots \right) \;,$$

quantitatively confirms our suspicions that the Incomplete Gamma Function is exponentially small compared to the n!, and it tells us precisely what "largish" means. As is true with all asymptotic series, this series is convergent for no value of τ , (which is why we use a ~ instead of an =) but, instead the terms in parenthesis will successively decrease in size until one reaches the N-th term, say, and then they will increase in size beyond this term. Stopping just short of this N-th term gives us the best approximation we can attain for the Incomplete Gamma Function, and the N-th term indicates how big our margin of error is. In practice for this process to yield useful results we will need $\tau_0 \gg n$, otherwise we arrive at the magic N-th term rather quickly. This tells us that for this whole procedure to make sense, we need $S^{(n)}$ to get small with n since, were this not the case, eventually no matter how big τ_0 is we would reach an n in our expansion when it is no longer big enough for the integral to be n!.

Thirdly, we have to repeat this over and over again for all the \mathbf{y} 's, \mathbf{n} 's and ν 's of interest, which are not even a countable sets of objects that you could list. So, to be honest, all this effort has only gotten us I at one place (well to be fair, we do have the solution at all the places *behind* it at retarded times, and in front of it at advanced times with a little ingenuity) in one direction, for one frequency and one time. *Ouch!* At this rate we are going nowhere fast.

5. Solving the Transfer Equation. Part 2

To speed things up, so speak, there are essentially *two* distinct directions one could go at this point. In real world applications, you'll find that the opacity and the source function are usually extremely complicated functions of position, time, frequency, and photon wavenumber. So accepting the inevitability of this fact, the next steps are to devise cunning, robust and efficient numerical algorithms to obtain sufficiently accurate solutions of the transfer equation for a given application. This is an entire course of study, and if this is where you see your future, the bibliography will give some places to start with this approach.

The second approach, which we pursue here, is to adopt some idealized, but not completely ludicrous, parameterized behaviors for these two functions, χ and S, and push the mathematics a little farther to understand the underlying physics of radiative transfer. This is often useful especially when used in concert with the previous direction.

Of the various approximations we might consider, taking the interaction of the radiation with the underlying material to be *isotropic*, or independent of photon propagation direction \mathbf{n} is powerful, and, often is not a bad approximation in many astrophysical circumstances. This could be on account of the fact that the scatterers/absorbers are really spherically symmetric, or because there is no preferred orientation for anisotropic scatter/absorbers. Both will give the same result, albeit, perhaps with different numerical values.

Andy yet—sigh—this is still not sufficient to get us all the **y**'s, **n**'s in one handy formula. We must also specialize to static planar, or spherical geometries. They apply to real situations where spatial variations in two (transverse) spatial coordinates are on very much larger scales when compared to the variations in the third, orthogonal, direction. As such geometries are frequently reasonable approximations to actual astrophysical systems, it makes sense to study them and be familiar with their properties. Cylindrical geometry is also amenable to analysis, but to be honest it is hard to dream up astrophysical situations which closely resemble cylinders. Spherical geometry often looks locally planar, except for situations in which the net optical depth across the sphere is less than or equal a number of order unity—this rarely occurs in astrophysical situations, although geophysically, clouds can often have this property as they are forming.

So we will confine our attention to Cartesian slab geometry, for which the time-independent transfer equations (at any given frequency ν , which we continue to suppress in the notation) is

$$\mu \frac{\partial I}{\partial z} + \chi I = \chi S$$

where $\mu = \mathbf{n} \cdot \hat{\mathbf{e}}_z$ and just to be definite, we shall assume that the slab of material lives somewhere in the half-space $z \leq 0$. If $\chi = \chi(z)$ is some function of depth into the slab, then as before, it behoves us to mathematically incorporate this variation into an optical depth coordinate measured from the surface z = 0 into the slab according to

$$au(z) = \int_{z}^{0} ds \ \chi(s) \ , \ \ {\rm for} \ z \leq 0 \ ,$$

to obtain

$$\mu \frac{dI}{d\tau} = I - S \; ,$$

which, save for the factor of μ , is precisely the same equation we encountered in the previous section. On the other hand, if χ can be treated as a constant independent of position, then in the next section we will indicate how a more powerful approach can be adopted! Astrophysically, the gravitational stratification of systems generally precludes any sensible attempt to adopt a position-independent opacity. Geophysically, clouds, again, will often behave in this fashion.

The innocuous factor of μ makes a huge impact! For $\mu > 0$, we integrate *inward* into the slab from τ until we reach the back surface of the slab at some optical depth, say τ_0 , to obtain

$$I(\tau,\mu) = e^{\frac{\tau-\tau_0}{\mu}} \ I_0 + \frac{1}{\mu} \int_{\tau}^{\tau_0} d\tau' e^{\frac{\tau-\tau'}{\mu}} \ S[z(\tau')] \ , \ \ 1 \ge \mu > 0 \ ,$$

where $I_0(z_0, \mu)$ is the specific intensity incident, or shining upon, the back of the slab located at $z = z_0$ (remember z_0 is negative by out conventions).

Conversely for $\mu < 0$ we integrate *outward* from the point τ to the (other) surface of the slab at z = 0

$$I(\tau,\mu) = -\frac{1}{\mu} \int_0^{\tau} d\tau' e^{\frac{\tau-\tau'}{\mu}} S[z(\tau')] , \quad -1 \le \mu < 0 .$$

We can also incorporate some intensity shining inward on this face of the slab as well, if that is desirable, in an obvious fashion. We will skip it here. A few subtleties are worth noting here. First, the point $\mu = 0$ must be handled with some measure of care through a limiting process from both positive and negative μ hemispheres. Second, if we are given the source function as it varies with depth into the slab, then to affect the integration we need to transform it to the optical depth scale. Third, a finite value for τ_0 implies that the slab is of finite extent in z. A semi-infinite slab has $\tau_0 \to \infty$ and provided I_0 doesn't do anything silly we drop the first term in the expression for I when $0 < \mu \leq 1$. We'll carry on with the semi-infinite slab, so

$$I(\tau,\mu) = +\frac{1}{\mu} \int_{\tau}^{\infty} d\tau' e^{\frac{\tau-\tau'}{\mu}} S[z(\tau')] , \quad 1 \ge \mu > 0$$

This constitutes a complete solution to our problem, which can then be mapped back on to a physical z scale by inverting the optical depth. Since ν does not appear explicitly in the solution, we can incorporate any ν dependence we like into $\chi_{\nu}(z)$ and $S_{\nu}(z)$ using the same formulas.

Since S and χ are independent of μ , we can readily compute the moments

$$J(\tau) = \frac{1}{2} \int_{-1}^{1} d\mu \ I(\tau, \mu) \equiv \Lambda_{\tau}[S(\tau')]$$
$$H(\tau) = \frac{1}{2} \int_{-1}^{1} d\mu \ \mu \ I(\tau, \mu) \equiv \Phi_{\tau}[S(\tau')]$$
$$K(\tau) = \frac{1}{2} \int_{-1}^{1} d\mu \ \mu^{2} \ I(\tau, \mu) \equiv X_{\tau}[S(\tau')]$$

as functionals, or equivalently integrals, over $S(\tau)$ with different kernel functions of $(\tau - \tau')$. In other words, these are *convolution* integrals. This is not too surprising because $I(\tau, \mu)$ is itself a convolution integral with an exponential kernel.

Just like there are table of integrals, there are extensive tables of what the operators Λ_{τ} , Φ_{τ} and X_{τ} do to various $S(\tau)$'s. We provide some references in the bibliography. Use them, don't try to do all these integrations yourself.

Now comes the interesting complication that underscores an essential quality of radiating fluids. As we shall demonstrate in a subsequent chapter, *if* the radiation field *and* the material are in local thermodynamic equilibrium (LTE) at the *same* temperature T(z) in the slab, *then* two thing must be true. *First*, the source function must be equal to the Planck Function

$$S_{\nu}(z) = B_{\nu}[T] = \frac{2h\nu^3}{c^2} \frac{1}{\exp[h\nu/k_B T(z)] - 1} ,$$

where h is Planck's Constant, k_B is Boltzmann's Constant, and c is the speed of light. (This is an exceedingly powerful result!) *Second*, the mean intensity must equal the source function, which equals the Planck Function. In our slabgeometry, this means

$$J(\tau) = \frac{1}{2} \int_{-1}^{1} d\mu \ I(\tau,\mu) \equiv \Lambda_{\tau}[J(\tau')] ,$$

or equivalently

$$S(\tau) = \Lambda_{\tau}[S(\tau')] ,$$

implying that we are no longer free to specify the source function, but, in fact, we need to solve for it! This is a real game changer.

This convolutional integral equations is known as *Milne's Integral Equation*. And once this equation is solved, we can find $T(\tau)$ from the frequency-integrated Planck Function:

$$B[T] \equiv \int_0^\infty d\nu \ B_\nu[T] \equiv \frac{\sigma_R}{\pi} T^4 = \frac{2\pi^4 k_B^4}{15h^3 c^2} T^4$$

where σ_R is the Stefan-Boltzmann Constant.

Of course, this equation *has* been solved, indeed—and here's the rub—there are *entire books* devoted to determining the solution and documenting its behavior.

6. Solving the Transfer Equation. Part 3

Consider finally the situation we alluded to above, in which we may take χ to be a strict constant, say χ_0 just to avoid confusion. And as we did at the conclusion of the previous section, we shall assume LTE. Then directly from the transfer equation, one may derive

$$S(\mathbf{x}) = \chi_0 \int d\mathbf{x}' \; \frac{e^{-\chi_0 |\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|^2} \left[S(\mathbf{x}') + Q(\mathbf{x}') \right] \; ,$$

which is valid in *any* geometry! Notice that without the exponential factor, this looks somewhat like our solution to Poisson's Equation in Scene 1, except we have an additional power of $|\mathbf{x} - \mathbf{x}'|$ in the denominator and the exponential. The additional term $Q(\mathbf{x}')$ is included in case there are some externally supplied photons from sources that have nothing to do with LTE and the material. This term would represent say a lighthouse in a fog bank, and the *S* would describe the scattering of this light by the fog.

Whole books do not need to be written about this equation or how to solve it. This (three-dimensional) convolutional kernel has an analytic Fourier Transform and so the method of choice is to Fourier Transform this equation using the convolution theorem for Fourier Transforms. The transformed equation is algebraic.

7. Exercises

Exercise 1: THE CASIMIR EFFECT

Consider this highly-idealized, but hopefully, very illuminating problem. Two parallel infinitely conducting plates are initially separated by a distance L. In between these two plates is vaccuum, that is, no matter so $\rho = 0$. The plates have equal and opposite surface charge densities, σ [dimensions: esu cm⁻²] and so there is a uniform electric field **E** filling the gap between the plates. Both plates are at the same temperature T, and are in thermal equilibirum with a radiation field that fills the gap between the plates and which is also at the same temperature T as the plates. (A) The plates attract each other due to the electric field, and the radiation field exerts a pressure $\mathbb{P} = P(T)\mathbb{1}$ that tries to push the plates apart. Enforcing mechanical equilibrium, find a relationship between σ , L, and T, say

$$L_{\rm eq}(T,\sigma)$$

(B) Now suppose that you apply a force to hold the plates apart at a distance L greater than L_{eq} , while keeping the temperature and the surface charge density fixed. Taking this initial state to be in thermal and mechanical equilibirum, at t = 0 you let go of the plates. Assume that the back of the plates are thermally insulated. What is the final separation of the plates and the temperature of the radiation field? How long does it take to reach this final state?

(C) A purely quantum electrodynamic process is the so-called *Casimir Effect*. It states that the vaccuum fluctuations which take place in the gap between the plates will on average produce at attractive force between the plates even when there is no charge on the plates and no electromagnetic field between them! The Casimir force per unit area is

$$hc\pi$$

$$480L^4$$
 ,

where, h is Planck's constant. How close would the two charged plates have to be in order that the neglected quantum electrodynamic effects embodied in the Casimir Force would invalidate our purely classical treatment of this problem?

Exercise 2: <u>STIMULATED EMISSION</u> If you muse over the Planck Function

$$B_{\nu}[T] = \frac{2h\nu^3}{c^2} \frac{1}{\exp[h\nu/k_B T] - 1} \; ,$$

for awhile, and think back to the discussion in the very beginning of the *Opera* on photon occupation numbers and Bose-Einstein statistics, it will strike you that the factor

$$\frac{2h\nu^3}{c^2}$$

has the same dimension as I_{ν} . This "threshold" specific intensity is associated with the relative importance of *stimulated emission* relative to *spontaneous emission*.

(A) In Pomraning $[\mathbf{P} \ \mathbf{5}]$, pp. 44-7, you'll find the equation of radiative transfer written as:

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mathbf{n}\cdot\nabla I_{\nu} = \overline{\eta}_{\nu}\left[1 + \frac{c^2}{2h\nu^3}I_{\nu}\right] - \overline{\chi}_{\nu}I_{\nu} \ ,$$

where $\overline{\eta}_{\nu}$ and $\overline{\chi}_{\nu}$ are the emissivity and the opacity, corrected for stimulated emission—stimulated emission being described by the second term in square brackets. Verify that

$$\overline{\eta}_{\nu} = \chi_{\nu} B_{\nu}[T] \left(1 - e^{-\frac{n\nu}{k_B T}} \right)$$
$$\chi_{\nu} = \overline{\chi}_{\nu} \left(1 + \frac{c^2}{2h\nu^3} B_{\nu}[T] \right)$$

are the necessary LTE relationships between the emissivity and opacity for spontaneous emission/absorption, and their counterparts which take account of stimulated emission/absorption, in order that $I_{\nu} = B_{\nu}[T]$ causes the right side of Pomraning's radiative transfer equation to vanish.

(B) Discuss the limiting behavior of the right side of Pomraning's equation when $h\nu/k_BT \ll 1$ and there are large numbers of photons in a unit phase-space cell of size h^3 . Does this provide any insight on when to use Maxwell's Equations instead of the equation of radiative transfer in describing the electromagnetic field? What is the situation when $h\nu \gg k_BT$?

(C) Usually, stimulated emission is presented in the context of the emission and absorption of radiation in very narrow spectral windows associated with specific atomic transitions, where one encounters the famous Einstein A and B coefficients. This is covered very nicely in the lecture notes by Rob Rutten, cited below. The generalization of these ideas to continuum processes is due to Milne.

Exercise 3: DIFFUSION APPROXIMATION

Substitute the Taylor series expansion

$$S(\tau') = S(\tau) + (\tau' - \tau) \frac{dS}{d\tau} + (\tau' - \tau)^2 \frac{1}{2!} \frac{d^2S}{d\tau^2} + \cdots ,$$

into the two exprssions for the specific intensity

$$\begin{split} I(\tau,\mu) &= +\frac{1}{\mu} \int_{\tau}^{\infty} d\tau' e^{\frac{\tau-\tau'}{\mu}} \; S(\tau') \;, \quad 1 \ge \mu > 0 \;, \\ I(\tau,\mu) &= -\frac{1}{\mu} \int_{0}^{\tau} d\tau' e^{\frac{\tau-\tau'}{\mu}} \; S(\tau') \;, \quad -1 \le \mu < 0 \;, \end{split}$$

valid for the semi-infinite slab with no light shining on it from the outside. (A) Convince yourself that

$$I(\tau,\mu) = \sum_{k=0}^{\infty} \mu^k \frac{d^k S}{d\tau^k} \left[1 - \theta(-\mu) \frac{1}{k!} \Gamma(1+k;-\tau/\mu) \right] ,$$

where $\theta(x)$ is the Heaviside step function which is 1 when x > 0 and 0 when x < 0, and $\Gamma(z, \zeta)$ is the Incomplete Gamma Function. (B) Now use this result to show

$$J = S + \frac{1}{3}\frac{d^2S}{d\tau^2} + \cdots$$
$$H = \frac{1}{3}\frac{dS}{d\tau} + \frac{1}{5}\frac{d^3S}{d\tau^3} + \cdots$$
$$K = \frac{1}{3}S + \frac{1}{5}\frac{d^2S}{d\tau^2} + \cdots$$

and comment on the content of the " \cdots "—how close can one be to the surface $\tau = 0$ before all of this goes terribly wrong?

(C) The diffusion approximation usually gets additional help because each higher τ -derivative of S is generally much smaller than the previous one. The *Eddington Approximation* amounts to asserting

$$K = \frac{1}{3}S$$

and neglecting the higher order derivatives. Use this result in the moment equations to arrive at a diffusion equation for the energy density in the radiation field:

$$\frac{\partial E}{\partial t} = \nabla \left(\frac{c}{3\chi} \nabla E \right) + \chi [4\pi S - cE] \; .$$

What is **F**?

Exercise 4: STILL MORE MURAM

Go back to Figure 6 provided at the end of Scene 2, and notice that the average temperature declines linearly, or very nealy so, with altitude in the lower half of the simulation.

(A) Use

$$S(\tau) = \frac{\sigma_R}{\pi} T^4$$

in your expression for \mathbf{F} above to obtain

$$F_3 = -\frac{16\sigma_R T^3}{3\chi} \frac{\partial T}{\partial x_3} \; .$$

(B) Estimate $d\langle T \rangle/dx_3$ from Figure 6, and use the fact that MURaM has

$$\langle F_3 \rangle \approx 6.3 \times 10^{10} \text{ erg cm}^{-2} \text{ sec}^{-1}$$

to say something about $\langle \chi \rangle$, which is the inverse of the photon mean-free-path. (C) Now take a gander at Figure 10. Where the mean temperature has a linear gradient, we can safely assume [as you probably did in part (B)] that

$$\langle T^n \rangle \approx \langle T \rangle^n$$

But this is certainly not the case in the upper one-third of the computational domain. What is happening here?

8. Further Reading

Radiative transfer as a subject has somewhat fallen out of favor in modern physics and astrophysics curricula, perhaps, in part, because it has the reputation of being terribly dry, very classical in outlook, and incredibly parsimonious in yielding useful analytic results or examples of pedagogical value in teaching. Today, radiative transfer, while an essential component of lots of investigations invariably enters as a large numerical "black box" of a code, based on shortcharacteristics, lambda-interation and a host of clever schemes to increase the pace of convergence. With these caveats in mind, and in keeping with my leitmotif of "if you could only own one book, or have one book on a desert island", my offering here is \star [M 4] Dimitri Mihalas, Stellar Atmospheres, (San Francisco, CA: W.H. Freeman and Company; 1970), xiv+463.

This is the first edition. The distinguished astrophysicist, Ivan Hubeny, has recently released a much enlarged and heavily revised n-th revision as

[HM 1] Ivan Hubeny & Dimitri Mihalas, <u>Theory of Stellar Atmospheres</u>. An Introduction to Astrophysical Non-Equilibrium Quantitative Spectroscopic Analysis, (Princeton, NJ: Princeton University Press; 2015), xvi+923,

which definitely bridges the gap between the classical and modern incarnations of radiative transfer, but at the price of losing some of the clarity and elegance of the more compact first edition.

Equally as useful, albeit from a very very different perspective, is the truly amazing contribution by

*[**RL 1**] George B. Rybicki & Alan P. Lightman, <u>Radiative Processes in Astro-</u>physics, (New York, NY: John Wiley & Sons; 1979), xv+382,

so, in fact, take *both* of these books if you are planning on spending a lot of time on a deserted island.

A very very close second (third?) to the Mihalas/Rybicki-Lightman combo, and much cheaper any way you look at it, is Rob Rutten's lecture notes which are available as a 275 page pdf on-line

https://www.staff.science.uu.nl/~rutte101/rrweb/rjr-edu/coursenotes/

Of the various classical treatises on radiative transfer, the following four are worth the effort to find in a used bookstore,

*[**K** 2] V. Kourganoff (with I.W. Busbridge), <u>Basic Methods in Transfer Problems.</u> <u>Radiative Equilibrium and Neutron Diffusion</u>, (Oxford, UK: Clarendon Press; 1952), xv+281,

[S 5] V.V. Sobolev, <u>A Treatise on Radiative Transfer</u>, (Princeton, NJ: D. Van Nostrand Company; 1963), xi+319,

[**G** 4] R.M. Goody, Atmospheric Radiation. I. Theoretical Basis, (Oxford, UK: Clarendon Press; 1964), xi+436,

*[**D** 2] B. Davison (with J.B. Sykes), <u>Neutron Transport Theory</u>, (Oxford, UK: Clarendon Press; 1957), -+450.

For an authoritative discussion of the Milne Equation for the plane-parallel problem, see

[H 2] Eberhard Hopf, Mathematical Problems of Radiative Equilibrium, (Cambridge, UK: Cambridge University Press; 1934), viii+105,

[**B** 4] I.W. Busbridge, <u>The Mathematics of Radiative Transfer</u>, (Cambridge, UK: Cambridge University Press; 1960), x+143.

Of course, Shu [S 1] and Mihalas & Mihalas [MM 1] are also very good sources for this same material. Finally, here we should also record the other monographs and treatises devoted to radiation hydrodynamics. All of them are worth looking at for the different perspectives they take and various approaches to the same basic issues. In no particular order, $\star [{\bf P}~{\bf 5}]$ Gerald C. Pomraning, The Equations of Radiation Hydrodynamics, (Mineola, NY: Dover Publications; 2005), x+286,

*[**P** 6] Shih-I Pai, <u>Radiation Gas Dynamics</u>, (New York, NY: Springer-Verlag; 1966), viii+229,

*[C 3] John Castor, <u>Radiation Hydrodynamics</u>, (Cambridge, UK: Cambridge University Press; 2004), xii+355.

9. Appendix A: Spherical Geometry

The derivation of the transfer equation in spherical geometry involves some subtleties which are handled very nicely in Chapter II of Pomraning [**P** 5]. In addition to the spherical coordinates r, θ, ϕ , we define $\mu = \mathbf{n} \cdot \mathbf{x}/|\mathbf{x}|$ as the cosine of the polar angle between the photon direction of propagation \mathbf{n} and the radial unit vector $\hat{\mathbf{e}}_r = \mathbf{x}/|\mathbf{x}|$. The second azimuthal propagation angle, φ , is the angle between the projection of \mathbf{n} onto the two-dimensional tangent plane perpendicular to \mathbf{x} at \mathbf{x} , and some fiducial direction lying in this plane, which could be north, south, east or west, take your pick, but remain consistent.

We shall be content simply to quote the result here:

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mu \frac{\partial I_{\nu}}{\partial r} + \frac{\sqrt{1-\mu^2}}{r} \left(\cos\varphi \frac{\partial I_{\nu}}{\partial \theta} + \frac{\sin\varphi}{\sin\theta} \frac{\partial I_{\nu}}{\partial \phi}\right) \\ + \frac{1-\mu^2}{r}\frac{\partial I_{\nu}}{\partial \mu} - \left(\frac{\sqrt{1-\mu^2}\sin\varphi\cot\theta}{r}\right)\frac{\partial I_{\nu}}{\partial \varphi} = \eta_{\nu} - \chi_{\nu}I_{\nu} ,$$

for the specific intensity $I_{\nu}(r, \theta, \phi, t; \mu, \varphi)$. For spherical symmetry we may take $\partial/\partial \theta = \partial/\partial \phi = \partial/\partial \varphi = 0$, resulting in

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mu \frac{\partial I_{\nu}}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_{\nu}}{\partial \mu} = \eta_{\nu} - \chi_{\nu}I_{\nu} ,$$

which is much less daunting in appearance, for a simple function $I_{\nu}(r, t; \mu)$.

Of course, you do not want to even *think* about what this might look like in oblate spheroidal coordinates.

10. Appendix B: A Short Table of Lambda Operators

The Lambda Operator,

$$F(\tau) = \Lambda_{\tau}[f(t)] \equiv \frac{1}{2} \int_0^\infty dt \ f(t) E_1(|t-\tau|)$$

is an operator or a mapping that assigns to every "integrable" function f(t) defined for $t \in [0, \infty)$ another function F(t). The vector space of integrable functions so defined on this interval is an example of an infinite dimensional vector space, or simply a function space. Alternatively, the Lambda Operator can be thought of an an integral transform.

The kernel of the integral transform, is the n = 1 incarnation of a family of functions

$$E_n(z) \equiv \int_1^\infty dt \ t^{-n} e^{-zt}$$

called the *Exponential Integral Function*, with an uncanny resemblence to the Gamma Function

$$\Gamma(z) \equiv \int_0^\infty dt \ t^{z-1} e^{-t}$$

which interpolates the factorial function. But that's about where the similarities end. However, the *Incomplete Gamma Function* introduced in $\S4$

$$\Gamma(z,\zeta) \equiv \int_{\zeta}^{\infty} dt \ t^{z-1} e^{-t} \ .$$

is more directly related.

The members of this family satisfy a simple recurrence relation

$$E_n(z) = \frac{1}{n-1} \left[e^{-z} - z E_{n-1}(z) \right], \ n = 2, 3, 4, \cdots$$

One also has

$$E_1(z) = -\log z - \gamma - \sum_{k=1}^{\infty} \frac{(-z)^k}{k k!}$$

where γ , like e and π is a fundamental mathematical constant, called Euler's constant. It is usually defined through a limiting process as

$$\gamma \equiv \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right) = 0.5772 \ 15664...$$

similar to

$$e \equiv \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = 2.7182 \ 81828...$$

(can you find a similar limit expression for π ?). Notice that $E_1(z)$ diverges logarithmically as $z \to 0$, but this is no cause for concern because such a divergence is integrable.

The series for $E_1(z)$ converges for all values of z, but, for largish z it converges too slowly to be of any practical use. Just as we did for the Incomplete Gamma Function in §4, we can find a *asymptotic* series that provides a much better estimate at large z,

$$E_n(z) \sim \frac{e^{-z}}{z} \left(1 + \frac{n}{z} + \frac{n(n+1)}{z^2} + \frac{n(n+1)(n+2)}{z^3} + \cdots \right) \;.$$

This series converges for no values of z, but, if $|z| \gg n$, the terms will decrease in magnitude to a very small value before they begin their inexorable rise and blow the sum out of the water. The sum up to that small value is our best available approximation to $E_n(z)$ from the asymptotic series.

With these preliminaries out of the way, on to the table:

$$\Lambda_{\tau}[1] = 1 - \frac{1}{2}E_2(\tau)$$

$$\Lambda_{\tau}[t] = 1! \left[\frac{1}{1} \frac{\tau}{1!} + \frac{1}{2} E_3(\tau) \right]$$
$$\Lambda_{\tau}[t^2] = 2! \left[\frac{1}{3} + \frac{1}{1} \frac{\tau^2}{2!} - \frac{1}{2} E_4(\tau) \right]$$
$$\Lambda_{\tau}[t^3] = 3! \left[\frac{1}{3} \frac{\tau}{1!} + \frac{1}{1} \frac{\tau^3}{3!} + \frac{1}{2} E_5(\tau) \right]$$
$$\Lambda_{\tau}[t^4] = 4! \left[\frac{1}{5} + \frac{1}{3} \frac{\tau^2}{2!} + \frac{1}{1} \frac{\tau^4}{4!} - \frac{1}{2} E_6(\tau) \right]$$
$$\Lambda_{\tau}[t^5] = 5! \left[\frac{1}{5} \frac{\tau}{1!} + \frac{1}{3} \frac{\tau^3}{3!} + \frac{1}{1} \frac{\tau^5}{5!} + \frac{1}{2} E_7(\tau) \right]$$
$$\Lambda_{\tau}[t^6] = 6! \left[\frac{1}{7} + \frac{1}{5} \frac{\tau^2}{2!} + \frac{1}{3} \frac{\tau^4}{4!} + \frac{1}{1} \frac{\tau^6}{6!} - \frac{1}{2} E_8(\tau) \right]$$

and you can now fill in the entry for an arbitrary power of t—our brains are so adept at seeing patterns! Also useful are

$$\Lambda_{\tau}[e^{-at}] = \frac{e^{-a\tau}}{2a} \left[\log \frac{|a+1|}{|a-1|} - E_1(\tau - a\tau) \right] + \frac{1}{2} E_1(\tau) ,$$

and

$$\Lambda_{\tau}[q(t)] = q(\tau) - \frac{1}{2}E_3(\tau)$$

where $q(\tau)$ is the *Hopf Function*. Therefore any multiple of

$$\tau + q(\tau) = \Lambda_{\tau}[t + q(t)]$$

is a solution of Milne's Integral Equation!

Armed therefore with these results for arbitrary powers and exponentials, you can build your own favorite source function $S(\tau)$ for the slab and compute the mean intensity $J(\tau)$ at any location!

It is worth noting in passing that the other two operators are also expressible in terms of the exponential integral functions:

$$\frac{1}{2} \Phi_{\tau}[f(t)] \equiv \int_{\tau}^{\infty} dt \ f(t) E_2(t-\tau) - \int_0^{\tau} dt \ f(t) E_2(\tau-t) \ ,$$
$$\frac{1}{2} X_{\tau}[f(t)] \equiv \int_0^{\infty} dt \ f(t) E_3(|t-\tau|) \ .$$

Tables for these operators can be found in Kourganoff [K 2].

11. Appendix C: Stokes Polarimetry

The extension of the transfer equation to handle the polarization of photons requires replacing the scalar specific intensity, $I_{\nu}(\mathbf{x}, t; \mathbf{n})$ by a vector with 4 components of which the first entry is just I_{ν} :

$$\mathbf{I}_{\nu} = (I_{\nu}, Q_{\nu}, U_{\nu}, V_{\nu}) ,$$

which is usually referred to as the *Stokes Vector* for polarized light. The functions Q_{ν} and U_{ν} measure the degree of linear polarization, while V_{ν} describes the circular polarization. The four quantities that appear in \mathbf{I}_{ν} are often referred to as the *Stokes Parameters*, and they satisfy the equation

$$I_{\nu}^2 \ge Q_{\nu}^2 + U_{\nu}^2 + V_{\nu}^2$$
.

Recall from the preamble to this *Opera* that we related the mean intensity to the photon distribution function by

$$I_{\nu}(\mathbf{x},t;\mathbf{n}) = \sum_{\pm 1} \frac{h^4 \nu^3}{c^2} f_{\pm 1}(\mathbf{x},t;\mathbf{n},\nu) ,$$

where the sum runs over the two spin-states of the photon. In the same fashion

$$V_{\nu}(\mathbf{x},t;\mathbf{n}) = \sum_{\pm 1} \pm \frac{h^4 \nu^3}{c^2} f_{\pm 1}(\mathbf{x},t;\mathbf{n},\nu)$$

is the difference between the two spin states which leads to a net right or left hand circlular polarization. Note that some authors will use \mp instead of \pm .

The radiation field is unpolarized when $Q_{\nu} = U_{\nu} = V_{\nu} = 0$, therefore it is always possible to partition a radiation field locally into unpolarized and completely polarized components as follows

$$\mathbf{I}_{\nu} = (I_{\nu} - \sqrt{Q_{\nu}^2 + U_{\nu}^2 + V_{\nu}^2}, 0, 0, 0) + (\sqrt{Q_{\nu}^2 + U_{\nu}^2 + V_{\nu}^2}, Q_{\nu}, U_{\nu}, V_{\nu}) \ .$$

One can therefore regard the quantity

$$\frac{\sqrt{Q_{\nu}^2 + U_{\nu}^2 + V_{\nu}^2}}{I_{\nu}}$$

as measuring the degree or fractional net polarization of the radiation field.

The equation of transfer generalizes in an appropriate manner, with the opacity now becoming a 4x4 matrix which permits the mixing of the four Stokes Parameters:

$$\frac{1}{c}\frac{\partial \mathbf{I}_{\nu}}{\partial t} + \mathbf{n} \cdot \nabla \mathbf{I}_{\nu} = \boldsymbol{\eta}_{\nu} - \mathbb{X}_{\nu} \cdot \mathbf{I}_{\nu} = \mathbb{X}_{\nu} \left(\mathbf{S}_{\nu} - \mathbf{I}_{\nu} \right)$$

The off-diagonal elements of \mathbb{X}_{ν} are generated principally by large scale, quasistatic, magnetic fields, which split degenerate atomic energy levels and induce both linear and circular polarization upon the emitted and scattered photons from these levels. Indeed, it is by just such processes that we infer the existence of magnetic fields in astrophysical systems.

12. Appendix D: Radiative Transfer in a Spherical Cloud

Almost every book on radiative transfer treats the static planar problem for a slab of material. So just to be different, and difficult, we'll do the spherical problem! Naturally, we'll want to employ spherical coordinates (r, θ, ϕ) for our three-dimensional Euclidean space, and will seek spherically symmetric solutions for which all quantities of interest, depend only upon the radial coordinate, $r \equiv |\mathbf{x}|$. We'll also look for static solutions, and invoke isotropy for the opacity and the source function. The frequency remains a harmless (possibly binned) parameter, so we'll just drop it from our notation. This leaves the direction of photon propagation **n**. Consistent with spherical symmetry, the very best we can hope for in terms of simplification is that the specific intensity at any radius r depends only upon the angle, ϑ , between **n** and **x**:

$$\cos\vartheta = \frac{\mathbf{n}\cdot\mathbf{x}}{|\mathbf{x}|} \equiv \mu \; .$$

So, we seek a $I(r; \mu)$, based upon a prescription for $\chi(r)$ and S(r), which we shall assume vanish identically outside of a sphere of radius r = R. The sphere could be a cloud or a star. How hard can that be?

The equation of transfer (omitting the time derivative!) is now:

$$\mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} + \chi I = \chi S ,$$

which, at first blush, seems slightly unexpected. One might reasonably ask where the heck the $\partial/\partial\mu$ emerged from when the original equation only had a ∇ in it! The answer is that this comes from the fact that even if a photon interacts with nothing, a photon changes its value of μ according to our definition, as it moves in a straight line offset by some distance from the origin. Indeed, any photon that does not pass directly through the origin must approach $\mu = \pm 1$ as it recedes to infinity. The transfer equation must know that, of course and it must ensure that $I(r; \mu)$ becomes strongly peaked in the $\mu = \pm 1$ directions as $r \to \infty$. And that, precisely, is what this additional term is there to do. The analogous equation for the slab is identical to this equation but with the second term on the left side absent, and with r and μ interpreted appropriately for Cartesian slab geometry.

We have by now accumulated so much experience solving first order partial differential equations that you are probably already figuring out the integrals of the motion (hint: there are three, not including the frequency), and what new variables $(r, \mu) \rightarrow (s, \eta)$ should be used to reduce this to our canonical expression

$$f(s,\eta)\frac{\partial I}{\partial s} + \chi I = \chi S$$
.

But before we do that, there are a few things we can determine in advance about our solution without doing any difficult mathematics. For example, Iat the center of the star/cloud must be isotropic with the same number of photons per unit time traversing every solid angle no matter what the direction. Likewise, at the very surface of a cloud/star $I(R, \mu)$ must vanish for $-1 \le \mu \le 0$ since the only source of photons (via S) is within the star/cloud proper. Indeed at r = R, I is not differentiable with respect to μ at $\mu = 0$, although it is continuous! Provided we restrict our attention to $r \ne R$, to avoid this poor μ behavior, if we average the transfer equation over all μ and integrate the second term by parts, we have

$$\frac{dH_r}{dr} + \frac{2}{r}H_r + \chi(r)J(r) = \chi(r)S(r) ,$$

which integrates formally to give

$$H_r(r) = \frac{1}{r^2} \int_0^r ds \ s^2 \chi(s) [S(s) - J(s)] ,$$

where we have made use of the fact that I(r = 0) is isotropic and therefore $H_r(0)$ must vanish. From this we may deduce that

$$\lim_{r \to 0} r \chi(r) [S(r) - J(r)] = 0 ,$$

and

$$\lim_{r \to R} \int_0^r ds \ s^2 \chi(s) [S(s) - J(s)] \ge 0 \ .$$

Multiplying the transfer equation by μ and averaging over μ gives

$$\frac{dK_{rr}}{dr} + \frac{3}{r}K_{rr} - J(r) + \chi(r)H_r(r) = 0.$$

So to find the actual solution, there is no avoiding the basic problem of solving for $I(r; \mu)$.

If you figured out that something akin to

$$\begin{split} r &= R\sqrt{s\eta} \ , \qquad \mu = \pm \sqrt{\frac{s-\eta}{\eta}} \ , \\ \eta &= \frac{r}{R}\sqrt{1-\mu^2} \ , \qquad s = \frac{r}{R}\frac{1}{\sqrt{1-\mu^2}} \ , \end{split}$$

is what is required, then award yourself points! Finally,

$$f(s,\eta) = 2\mu = \pm 2\sqrt{\frac{s-\eta}{\eta}},$$

which, as advertized, is a functions of s and η . We'll see shortly why there is a factor of 2 here.

The transfer equation indicates that η is a constant of the motion. So, if we wish to determine I at some specified r' and μ' , we know that along our *s*-integration $\eta = \eta'$ will be strictly constant. And this tells us that there is a unique relationship between a photon's radial position r and propagation direction μ

$$r = R\sqrt{s\eta'}$$
, $\mu = \pm \sqrt{rac{s-\eta'}{\eta'}}$,

so that when it ends up at r = r' (or equivalently s = s') it is propagating in the direction $\mu = \mu'$.

If we are computing $I(r', \mu')$ inside our spherical star/cloud, then $0 \le \eta' \le 1$, and, for a fixed value of η' , it must be the case that $\eta' \le s' \le (1/\eta')$. The only way $s' = \eta'$, is for $\mu' = 0$, and conversely, for $s' = 1/\eta'$, we must have r = R. In other words, the maximum value of s is at the surface of the star/cloud, and the minimum value of s is inside the star/cloud at the minimum radius of closest approach to the center of the star/cloud where $\mu = 0$.

So, set up your equations and integrate them backward and forward ds at fixed η , to obtain the equations for I in terms of S. Then compute the moments!

13. Appendix E: RMHD's 58 Terms

The energy density of the radiation field is

$$E = \frac{4\pi}{c}J$$

and the *energy flux* is

$$\mathbf{F} = 4\pi \mathbf{H}$$
.

The *energy exchange* term with the material is

$$\dot{\mathcal{E}}^{M \to R} - \dot{\mathcal{E}}^{R \to M} = \int_0^\infty d\nu \oint d\mathbf{n} \left[\eta_\nu - \chi_\nu I_\nu \right] \,.$$

The momentum density of the radiation field is

$$\frac{1}{c^2}\mathbf{F} = \frac{4\pi}{c^2}\mathbf{H}$$

and the momentum flux tensor for the radiation field is

$$\mathbb{P} = \frac{4\pi}{c} \mathbb{K} \ .$$

The momentum exchange term with the material is

$$\dot{\mathcal{P}}_i^{M \to R} - \dot{\mathcal{P}}_i^{R \to M} = \int_0^\infty d\nu \oint d\mathbf{n} \ n_i [\eta_\nu - \chi_\nu I_\nu] \ .$$

The radiation *field equation* is

$$\frac{1}{c}\frac{\partial I_{\nu}}{\partial t} + \mathbf{n}\cdot\nabla I_{\nu} = \eta_{\nu} - \chi_{\nu}I_{\nu} \ .$$