Act I. Scene 3: The Electromagnetic Fields

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1. Introduction

Besides gravity, electromagnetism is the only other force field that operates over macroscopic distances. Accordingly, it interacts with the material and is capable of exchanging momentum and energy with the fluid. These exchange processes are what we must quantify to carry out our program of developing a working description of a radiating magnetofluid.

In a certain sense, the large-scale electromagnetic fields and the radiation field, although they are treated separately, are in fact different aspect of a single phenomenon distinguished simply by the frequency of their temporal variability. Radiation is electromagnetic waves that of course span a wide range of frequencies from radio (starting at a few kHz [dimensions: $\sec^{-1}$]) to gamma rays (frequencies in excess of 10^{19} Hz). Quasi-static electromagnetic fields start out at zero frequency, of course. Naturally before one reaches several kHz the division between what is quasi-static large scale electromagnetic field, and what is a low-frquency 3 kHz radio wave with wavelength of order 100 km becomes problematic. In practice this quandary is often moot because there is little sensible power in this zone.

2. Maxwell's Equations

The electric $\mathbf{E}(\mathbf{x},t)$ and magnetic $\mathbf{B}(\mathbf{x},t)$ fields [dimensions: gm$^{1/2}$ cm$^{-1/2}$ sec^{-1} — equivalently, Statvolts/cm or Gauss, respectively] in an inertial frame of reference are provided by

$$\nabla \cdot \mathbf{E} = 4\pi\delta \ , \qquad c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ ,$$

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad c\nabla \times \mathbf{B} = 4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \ ,$$

where c is the speed of light (in vaccuum), $\delta(\mathbf{x},t)$ [dimensions: gm$^{1/2}$ cm$^{-3/2}$ sec^{-1} — equivalently esu/cm^3] is the density of electric charge and $\mathbf{J}(\mathbf{x},t)$ [dimensions: gm$^{1/2}$ cm$^{-1/2}$ sec^{-2} — equivalently esu/cm^2/sec] is the electric current density. Conservation of charge,

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

is a direct consequence of these equations.

Because there are apparently no magnetic charges anywhere in the universe the magnetic field is purley solenoidal. The electric field and the current density can in general have *both* irrotational and solenoidal components. This is what makes these equations so interesting.

The electric charge density is a source term for the irrotational component of the electric field, while the solenoidal component of the electric current density is a source for the magnetic field (and the solenoidal component of the electric field). Indeed, for time-independent sources, we can immediately write down

$$\mathbf{E}(\mathbf{x}) = -\nabla \int d\mathbf{x}' \frac{\delta(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} = -\int d\mathbf{x}' \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \delta(\mathbf{x}') \equiv -\nabla\phi$$

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c}\nabla \times \int d\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} = \frac{1}{c}\int d\mathbf{x}' \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \times \mathbf{J}(\mathbf{x}') \equiv \nabla \times \mathbf{A} \ .$$

The quantity $\mathbf{A}(\mathbf{x})$ [dimensions: gm$^{1/2}$ cm$^{1/2}$ sec^{-1} — Gauss cm] is called the vector potential and $\phi(\mathbf{x})$ [dimensions:gm$^{1/2}$ cm$^{1/2}$ sec^{-1} — Statvolts] is the electrostatic potential. The solenoidal component of $\mathbf{E}$ is zero. If electromagnetic fields are present which are generated by the presence of electric currents and charges outside of the domain of interest, then we may add to the right side of these equations any solutions of the homogeneous Maxwell Equations:

$$\nabla \cdot \mathbf{E} = 0 \ , \qquad c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ ,$$

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad c\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \ ,$$

which include, for example, photons (electromagnetic radiation).

In addition to the conservation of charge, there are two additional conservation laws which can be derived easily from the full set of Maxwell's Equations. The first expresses the conservation of momentum:

$$\frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \mathbb{M} = -\delta\mathbf{E} - \frac{1}{c}\mathbf{J} \times \mathbf{B} \ .$$

The momentum density per unit volume carried by the electromagnetic fields is proportional to the Poynting Flux [dimensions: gm sec^{-3}]

$$\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \ ,$$

and the (symmetric) Maxwell stress tensor [dimensions: gm cm^{-1} sec^{-2}] is

$$\mathbb{M} \equiv \frac{1}{8\pi}\left[\mathbb{1}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - 2(\mathbf{EE} + \mathbf{BB})\right] \ .$$

The term on the right side of this equation, which is the negative of the Lorentz Force, is the rate at which momentum per unit volume is exchanged with the material supporting the charge density and the electric currents. *It therefore follows that if the combined system of material and the electromagnetic fields conserves linear momentum, then:*

$$\frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \mathbb{M} = -\delta\mathbf{E} - \frac{1}{c}\mathbf{J} \times \mathbf{B} \ = -\rho\mathbf{a}^{EM} \ .$$

Recall that

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla \cdot \left(p\mathbb{1} - \mathbb{\sigma} + \rho\mathbf{uu}\right) = \rho(\mathbf{g} + \mathbf{a}^{EM} + \mathbf{a}^R) \ ,$$

thus by adding these two equations together we obtain

$$\frac{\partial}{\partial t}\Big(\rho\mathbf{u} + \frac{1}{c^2}\mathbf{S}\Big) + \nabla\cdot\Big(p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u} + \mathbb{M}\Big) = \rho(\mathbf{g} + \mathbf{a}^R) \ ,$$

which simplifies further to

$$\frac{\partial}{\partial t}\Big(\rho\mathbf{u} + \frac{1}{c^2}\mathbf{S}\Big) + \nabla\cdot\Big(p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u} + \mathbb{M} + \mathbb{G}\Big) = \rho\mathbf{a}^R \ ,$$

if and only if the fluid is self-gravitating! Recall from Scene 2 that

$$\mathbb{G} \equiv \frac{1}{8\pi G}(2\mathbf{g}\mathbf{g} - \mathbb{1}|\mathbf{g}|^2) \ , \qquad \mathbf{g} = -\nabla\Phi \ ,$$

is very similar in appearance, but of opposite sign, when compared with $\mathbb{M}$, and it has a single field $\mathbf{g}$ in lieu of the two electromagnetic fields, $\mathbf{E}$ and $\mathbf{B}$. Therefore $\mathbf{g}/\sqrt{G}$ has the same dimensions as $\mathbf{E}$ and $\mathbf{B}$—a useful observation if one contemplates a unified theory of gravity and electromagnetism!

This equation implies (*i*) that momentum per unit volume can be stored in both the material and the electromagnetic fields and (*ii*) that it is transported spatially by a series of symmetric tensors. The Lorentz Force

$$\rho\mathbf{a}^{EM} = \delta\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}$$

describes the process of momentum transfer between the electromagnetic fields and the material. It can have either sign depending upon whether the fields are doing work on the material or vice-versa. It vanishes when the is no net exchange of momentum between the two systems.

The second conservation law that may be obtained from Maxwell's Equations expresses the conservation of energy

$$\frac{\partial}{\partial t}\Big(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi}\Big) + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E} \ .$$

The $\mathbf{J}\cdot\mathbf{E}$ term describes how energy is transferred between the electromagnetic field and the material.

Although we have successfully obtained an expression for $\mathbf{a}^{EM}$ and have sufficient number of equations to describe the additional electromagnetic fields needed to deterimne $\mathbf{a}^{M}$, we have four new functions, δ and $\mathbf{J}$, coupled by one constraint

$$\frac{\partial\delta}{\partial t} + \nabla\cdot\mathbf{J} = 0 \ .$$

Therefore we need three additional equations to determine the source terms. And this, again, requires some knowledge of the microphysics of our fluid. Appied electric and magnetic fields can induce polarization charge densities and magnetization currents as well as conduction currents if there are free charges present. Like the viscosity, it is necessary to carefully examine the prevalent microphysics on a case by case basis to understand how to determine transport coefficients. We'll return to this problem a little later.

To wrap up this section it remains simply to write down the solutions of Maxwell's Equations when the charge density and the electric currents are time dependent. Toward this end, it proves helpful to work with time-dependent analogues of the electric potential and the vector potential that are selected to satisfy the radiation gauge condition

$$\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \ .$$

As before, the (solenoidal) magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$, however, since the electric field has both irrotational and solenoidal components we must take

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

to account for both. We then find

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\phi = -4\pi\delta(\mathbf{x}, t) \ ,$$

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}(\mathbf{x}, t) \ ,$$

wave equations for the scalar and vector potentials. Had we assumed a different gauge condition, we would have derived different, coupled, equations for ϕ and $\mathbf{A}$.

The solutions are

$$\phi(\mathbf{x}, t) = \int d\mathbf{x}' \frac{1}{|\mathbf{x}' - \mathbf{x}|}\delta(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|/c) \ ,$$

and

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c}\int d\mathbf{x}' \frac{1}{|\mathbf{x}' - \mathbf{x}|}\mathbf{J}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|/c) \ ,$$

with the property that the source terms are to be evaluated at a past, or *retarded*, time consistent with the fact that light propagates at the speed c.

3. Electromagnetic Radiation

Electromagnetic waves are solutions of

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\mathbf{A} = 0 \ ,$$

where $\nabla \cdot \mathbf{A} = 0$. The starting point for building a *photon* is to notice that the plane-wave

$$A_j^{\pm}(\mathbf{x}, t) = \exp i[\mathbf{k} \cdot \mathbf{x} \pm |\mathbf{k}|ct] \ ,$$

$j = 1, 2, 3$ is a solution of the wave equations, for any constant vector $\mathbf{k}$ [dimensions: cm^{-1}] provided only that $\mathbf{k} \cdot \mathbf{A} = 0$. In the remainder if this chapter, we'll reserve, $i = \sqrt{-1}$—elsewhere it will be a summation index. As our electromagnetic fields are *real*, this plane-wave actually contains two possible independent solutions

$$\cos[[\mathbf{k}\cdot\mathbf{x} \pm |\mathbf{k}|ct] + i\sin[[\mathbf{k}\cdot\mathbf{x} \pm |\mathbf{k}|ct] \ .$$

This plane-wave is not, however, a particularly good example of a photon, because it covers all of space and exists for all times. Photons should be localized in both space and time, while retaining a definite direction of propagation $\mathbf{n} \equiv \mathbf{k}/|\mathbf{k}|$ and frequency of oscillation $\omega \equiv c|\mathbf{k}| = 2\pi\nu$ [dimensions: rad sec^{-1}].

Since the wave equation is a linear equation, we are free to select the cosine or the sine solution, or any linear combination thereof. We are also free to add together as many plane waves as we wish. In otherwords, for any (six!) function $a_j^{\pm}(\mathbf{k})$ [dimensions: gm$^{1/2}$ cm$^{7/2}$ sec^{-1} — Gauss cm^4] which render the following integrals meaningful,

$$A_j(\mathbf{x},t) = \sum_{\pm} \frac{1}{(2\pi)^3} \int d\mathbf{k}\ a_j^{\pm}(\mathbf{k}) \exp i[\mathbf{k}\cdot\mathbf{x} \pm |\mathbf{k}|ct] \ ,$$

is *also* a solution of the wave equation provided only that

$$0 = \sum_{\pm} \int d\mathbf{k}\ k_j a_j^{\pm}(\mathbf{k}) \exp i[\mathbf{k}\cdot\mathbf{x} \pm |\mathbf{k}|ct] \ ,$$

where $d\mathbf{k} \equiv dk_1\ dk_2\ dk_3$. The prefactor of $1/(2\pi)^3$ is related to the fact that $|\mathbf{k}| = 2\pi/\lambda$, there λ is the wavelength of the radiation.

The initial condition at $t = 0$ is

$$A_j(\mathbf{x},0) = \sum_{\pm} \frac{1}{(2\pi)^3} \int d\mathbf{k}\ a_j^{\pm}(\mathbf{k}) \exp i\mathbf{k}\cdot\mathbf{x} \ .$$

Now there are many many other families of functions which are solutions of the linear wave equation, but our motivation for selecting plane-waves now becomes apparent. This last equation tells us that $A_j(\mathbf{x},0)$ is the (three-dimensional) Fourier Transform of the sum of the two $a_j(\mathbf{k})$, therefore by the Fourier Inversion theorem [see, for example, Appendix B §7 of these notes (not of this Scene!) for some background if necessary], we know that

$$a_j^{+}(\mathbf{k}) + a_j^{-}(\mathbf{k}) = \int d\mathbf{x}\ A_j(\mathbf{x},0) \exp[-i\mathbf{k}\cdot\mathbf{x}] \ .$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, this initial condition is sufficient only to provide the initial magnetic field. Since $\mathbf{E} = -c^{-1}\partial\mathbf{A}/\partial t$, we require an additional initial condition

$$\frac{\partial}{\partial t} A_j(\mathbf{x},0) = \sum_{\pm} \pm \frac{ic}{(2\pi)^3} \int d\mathbf{k}\ |\mathbf{k}|\ a_j^{\pm}(\mathbf{k}) \exp i\mathbf{k}\cdot\mathbf{x} \ .$$

Which implies

$$a_j^{+}(\mathbf{k}) - a_j^{-}(\mathbf{k}) = -\frac{i}{c|\mathbf{k}|} \int d\mathbf{x}\ \frac{\partial}{\partial t} A_j(\mathbf{x},0)) \exp[-i\mathbf{k}\cdot\mathbf{x}] \ .$$

So now we have enough information to determine all the little a's

$$a_j^{\pm}(\mathbf{k}) = \frac{1}{2}\int d\mathbf{x}\ \Big[A_j(\mathbf{x},0) \mp \frac{i}{c|\mathbf{k}|}\frac{\partial}{\partial t}A_j(\mathbf{x},0))\Big]\exp[-i\mathbf{k}\cdot\mathbf{x}]\ ,$$

in terms of the *initial* electric and magnetic fields at time $t = 0$.

An observation or two is in order here before we set about constructing a photon. *First*, the six functions $\mathbf{a}^{\pm}(\mathbf{k})$ provide the amplitude, phase and polarization of electromagnetic waves propagating in the $\mathbf{k}$ ('$+$' sign) and $-\mathbf{k}$ ('$-$' sign) directions with frequency $\omega = c|\mathbf{k}|$. Only five of these six functions are independent owing to the constraint $\nabla \cdot \mathbf{A} = 0$. *Second*, for each direction of propagation, the polarization state of the electromagnetic wave is encoded in the directionality of the vector $\mathbf{a}$. For example, a plane polarized wave with its electric field vector everywhere aligned with the x_1-axis is achieved by setting $a_2^{\pm}(\mathbf{k}) = a_3^{\pm}(\mathbf{k}) = 0$. Elliptic polarization of the electric field vector in the plane perpendicular to the x_3-axis requires only that $a_3^{\pm}(\mathbf{k}) = 0$, and so forth. *Third*, although it is conventional to think of the $\mathbf{a}^{\pm}(\mathbf{k})$ as functions of the real three-dimensional wave vector, $\mathbf{k}$, their construction via the Fourier Inversion theorem enables them to be rgarded more generally as functions of three *complex variables* k_j, $j = 1, 2, 3$. Where, for example, a given k is composed of real and imaginary parts according to $k \equiv \kappa + i\varkappa$, with both κ and $\varkappa$ being real variables and $i \equiv \sqrt{-1}$. In fact, in some circumstances the defining integral for $\mathbf{a}$ may not converge for purely real $\mathbf{k}$—it may exist only for some restricted domain in the complex plane. Finally, the vector properties of $\mathbf{a}$ determine the state of polarization of the electromagnetic plane-wave.

4. Building a (Perhaps Very Large) Photon

"Photons", here interpreted more loosely as localized electromagnetic fields with a predominant frequency/wavelength, are also solutions to the wave equation for $\mathbf{A}(\mathbf{x}, t)$. A photon is a *localized* wavepacket, or a carefully phased superposition, of plane-wave solutions concentrated about a definite oscillation frequency ω_0 [dimensions: rad sec^{-1}] and wavevector $\mathbf{k}_0$ [dimensions: cm^{-1}]. Crudely speaking, the spread of wavenumbers about $\mathbf{k}_0$ and frequencies about ω_0 that are required to construct the wavepacket satisfy the *uncertainty relations*

$$\Delta k \Delta x \approx 1\ , \qquad \Delta\omega\Delta t \approx 1$$

where Δx is the spatial extent of the wave packet at a fixed moment in time, and Δt is the temporal duration of the wavepacket at some fixed position in space. The plane-waves of the last section have $\Delta k = \Delta\omega = 0$ and therefore fill all of space and exist for an eternity.

To illustrate the construction process while keeping the notation reasonable, we'll work with one of the three components of $\mathbf{A}$—the other two proceed in the same fashion—and one direction of propagation (say '$-$'). Dropping superfluous subscripts and superscripts we have

$$A(\mathbf{x}, t) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\ a(\mathbf{k})\ \exp i[\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|ct]\ ,$$

where, of course,

$$a(\mathbf{k}) = \int d\mathbf{x}\ A(\mathbf{x}, 0) \exp[-i\mathbf{k} \cdot \mathbf{x}] \ .$$

A particularly useful starting point for building a wavepacket is a sinusoidal spatial oscillation with a Gaussian envelope:

$$A(\mathbf{x}, 0) = A_0 \exp\left(i\mathbf{k}_0 \cdot \mathbf{x} - \frac{1}{2} \frac{|\mathbf{x}|^2}{L^2} \right) + c.c.\ ,$$

for some specified values of the constants A_0, L (dimensions: cm) and $\mathbf{k}_0$ (dimensions: cm^{-1}). Here, the expression *c.c.*, denotes the complex conjugate that sends $i \to -i$. The second term serves to localize the disturbance to a ball of radius L at the origin (a constant $\mathbf{x}_0$ could be subtracted from $\mathbf{x}$ if it was advantageous to localize the disturbance elsewhere), and the first term selects a particular wave vector $\mathbf{k}_0$ for the photon. The (possibly) complex constant A_0, when combined with similar expressions for the other components of the full vector $\mathbf{A}$ sets the polarization properties and the phase. However, any other initial condition with these basic properties can be used as the initial shape and location of the wavepacket. So

$$a(\mathbf{k}) = A_0 \int d\mathbf{x}\ \exp\left[i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x} - \frac{1}{2} \frac{|\mathbf{x}|^2}{L^2} \right] + c.c.$$

The integrals are best done sequentially, after rotating the spatial integration coordinate system, $\mathbf{x} \to \mathbf{y}$, so that $\mathbf{k}_0$ points in the positive y_3 direction, say. If we let $k_\parallel$, denote the component of $\mathbf{k}$ that points in the direction of $\mathbf{k}_0$, and $\mathbf{k}_\perp$ the two components in the plane perpendicular to $\mathbf{k}_0$, we find

$$a(\mathbf{k}) = A_0 (L\sqrt{2\pi})^3 \exp\left(-\frac{L^2}{2} \left[|\mathbf{k}_\perp|^2 + (k_\parallel - |\mathbf{k}_0|)^2] \right] \right) + c.c.\ ,$$

or equivalently

$$a(\mathbf{k}) = A_0 (L\sqrt{2\pi})^3 \exp\left(-\frac{L^2}{2} \left[|\mathbf{k}|^2 + |\mathbf{k}_0|^2 - 2\mathbf{k} \cdot \mathbf{k}_0 \right] \right) +\ c.c.\ ,$$

As we noted near the beginning of this section, $a(\mathbf{k})$ is localized in a ball of radius L^{-1} about $\mathbf{k}_0$ in wavevector space. Therefore, the final desired result is

$$A(\mathbf{x}, t) = \frac{A_0 L^3}{(2\pi)^{3/2}} \int d\mathbf{k}\ \exp\left[i\mathbf{k} \cdot \mathbf{x} - i|\mathbf{k}|ct - \frac{L^2}{2} \left(|\mathbf{k}|^2 + |\mathbf{k}_0|^2 - 2\mathbf{k} \cdot \mathbf{k}_0 \right) \right] + c.c.$$

These three integrals are best carried out by making a change of variable from $\mathbf{k} \to \mathbf{p}$, where $\mathbf{p} = \mathbf{k} - \mathbf{k}_0$. Then

$$A(\mathbf{x}, t) = \frac{A_0 L^3}{(2\pi)^{3/2}} e^{i\mathbf{k}_0 \cdot \mathbf{x}} \int d\mathbf{p}\ \exp\left[i\mathbf{p} \cdot \mathbf{x} - i|\mathbf{p} + \mathbf{k}_0|ct - \frac{1}{2} L^2 |\mathbf{p}|^2 \right]\ +\ c.c.,$$

which is particularly simple in form and conveniently has the desired plane wave factor sitting outside of the integrals! However, the innocuous looking expression $|\mathbf{k}_0 + \mathbf{p}|$ prevents these integrals from being carried out analytically, even though this expression only differs from $|\mathbf{k}_0|$ by an amount L^{-1}. We therefore seek to approximate the exact solution by expanding this term in ascending powers of $\mathbf{p}/|\mathbf{k}_0|$ which is of order $1/(L|\mathbf{k}_0|) \approx 2\pi\lambda_0/L$, where λ_0 is the effective wavelength of the photon. Provided $L \gg \lambda_0$, we anticipate that only a few terms in this expansion will suffice to provide an excellent approximation to the exact result.

The desired expansion is straightforward, but somewhat tedious to carry out. Correct to order $(\mathbf{p}/|\mathbf{k}_0|)^5$ we obtain

$$\frac{|\mathbf{k}_0 + \mathbf{p}|}{|\mathbf{k}_0|} = \left(1 + \frac{p_\parallel}{|\mathbf{k}_0|} + \frac{|\mathbf{p}_\perp|^2}{2|\mathbf{k}_0|^2} - \frac{p_\parallel|\mathbf{p}_\perp|^2}{2|\mathbf{k}_0|^3} + \frac{p_\parallel^2|\mathbf{p}_\perp|^2}{2|\mathbf{k}_0|^4} + \frac{3p_\parallel|\mathbf{p}_\perp|^4}{8|\mathbf{k}_0|^5} - \frac{p_\parallel^3|\mathbf{p}_\perp|^2}{2|\mathbf{k}_0|^5} + \cdots\right)$$

where $p_\parallel$ is the component of $\mathbf{p}$ in the direction of $\mathbf{k}_0$, and $\mathbf{p}_\perp$ lies in the plane perpendicular to $\mathbf{k}_0$. It is a very intersting exercise to see how the approximation to $A(\mathbf{x}, t)$ changes as more terms are retained in this expansion. Traditionally, one retains the first *three* terms in this expansion, for which the integrals can be carried out analytically. This gives a wavepacket that begins to spread beyond L as the square root of the elapsed time t in the two transverse directions once $t \gg L^2|\mathbf{k}_0|/c$, but which retains its width L in the direction of propagation.

5. Summary

Although things appear to be very neat and tidy, unlike our causal headaches with Newtonian Gravity, all is not well with the classical electromagnetic field. Charges in motion create $\delta(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$, which in turn via Maxwell's Equations and their solutions, create electric and magnetic fields. These $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ fields in turn steer the charges through the Lorentz Force $\delta\mathbf{E} + (\mathbf{J} \times \mathbf{B})/c$. But, how does a charge interact with its *own* electric and magnetic fields? How can it distinguish its fields from the ones created by the charges around it? And all these electromagnetic waves that are being radiated off into space due to the acceleration of the charges, where do we take into account that this energy has to come from the charges themselves?

Imagine if you will a simple thought experiment. We fasten an electron at $x = +a$ and a positron at $x = -a$ and hold them there. It is not too hard to compute the electric field that exists throughout all of space in this static configuration, as well as how much force one has to apply to hold these two particles apart. Now, at $t = 0$ let go. What happens next? And what is the end state of this system and when is it achieved? Problems just like this caused physicists and mathematicians great consternation at the very beginning of the 20th Century.

6. Exercises

Exercise 1: BUILD YOUR OWN THEORY OF GRAVITY

Because there are time derivatives in Maxwell's Equations, light travels at a finite speed c, and the electromagnetic fields can store energy and momentum.

Newton's law of gravity, on the other hand, has no time derivatives, so there are no gravitational waves and we cannot store energy and momentum in the gravitational field itself. If you were to set about building a causal theory of gravity, you might be tempted to begin by replacing Poisson's Equation by

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2}\Big)\Phi = 4\pi G\rho(\mathbf{x}, t)\ .$$

This, however, can only be a part of the story, because the actual gravitational field is $\mathbf{g} = -\nabla\Phi$ just like $-\nabla\phi$ generates the irrotational component of the electric field $\mathbf{E}$.
(A) Assume the energy density in the gravitational field itself must be proportional to $|\mathbf{g}|^2$. What combination of G's and c's must you multiply this by to get something with the correct dimensions of erg cm^{-3}?
(B) Try to build a *scalar* theory of gravity that is consistent with the above wave equation for the potential. In otherwords, try find an equation for $\mathbf{g}$ with ρ as a source term that is consistent with the wave equation for Φ in the same fashion that Maxwell's Equations are equivalent to wave equations for ϕ and the three components of $\mathbf{A}$ subject to the gauge constraint

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0\ .$$

(C) Having run out of options in part (B), you can next try to build a *vector* theory of gravity by introducing a second gravitational field $\mathbf{h}$ and exploit the correspondence $\mathbf{g} \iff \mathbf{E}$, $\mathbf{h} \iff \mathbf{B}$, $\Phi \iff \phi$, and $\mathbf{a} \iff \mathbf{A}$. What does your $\mathbf{h}$ field do to matter?
(D) The next choice is a *tensor* theory of gravity, and, this is precisely where Einstein ended up after carrying out parts (B) and (C)!

Exercise 2: LOTS OF ENERGY EQUATIONS
We derived *an* energy equation for the electromagnetic fields in

$$\frac{\partial}{\partial t}\Big(\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi}\Big) + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}\ ,$$

but did we get the right one? Are there others?
(A) Start with the vector identity for *any* two vectors

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}\ ,$$

and then let $\mathbf{A}$ actually be the vector potential, and $\mathbf{B} = \nabla\times\mathbf{A}$ the magnetic field to convince yourself that

$$\frac{\partial}{\partial t}\Big(\frac{|\mathbf{B}|^2 - \mathbf{A}\cdot\nabla\times\mathbf{B}}{8\pi}\Big) - \nabla\cdot\frac{\partial}{\partial t}\frac{\mathbf{A}\times\mathbf{B}}{8\pi} = 0.$$

(B) One can obviously add any amount of this equation to the energy equation we derived in the text to get any number of energy equations, all of which, have the same term $-\mathbf{J}\cdot\mathbf{E}$ on the right side, but with different prescriptions for the

energy density and the energy flux. Which of all these possibilities should we pick, and why? Does it actually matter?

Exercise 3: A VARIETY OF WAVE PACKETS
In §3 we derived an expression for $|\mathbf{k}_0 + \mathbf{p}|$ valid for $|\mathbf{p}|/|\mathbf{k}_0| \ll 1$. First make sure that I did not screw up the algebra by deriving it for yourself!
(A) Then, explore how a wavepacket changes its character as more, or less, terms are retained in this expansion, by carrying out the three $d\mathbf{p}$ integrals back in the expression for $A(\mathbf{x}, t)$ at the bottom of page 7.
(B) Hint: *Sometimes* but not always (and this turns out to be the "not always" case) it is useful to convert $d\mathbf{p} = dp_1 dp_2 dp_3 = p^2 dp \sin\vartheta d\vartheta d\varphi$ into spherical polar coordinates, or cylindrical coordinates, $d\mathbf{p} = p_\perp dp_\perp d\varphi dp_\|$ to affect the integrations.
(C) The following four defnite integrals may (or may not, I make no promises here!) be helpful in part (A), but they will be helpful in general!

$$H(a,b) \equiv \int_0^\infty dk \exp(-ak^b) = \left(\frac{1}{a}\right)^b \Gamma\left(1 + \frac{1}{b}\right) ,$$

where $\Gamma(z)$ is the *Gamma Function*, defined by

$$\Gamma(z) \equiv \int_0^\infty dt\ t^{z-1} e^{-z}$$

and which interplolates the factorial function, $\Gamma(1+n) = n!$, when n is a non-negative integer.

$$I(a,b,c,d) \equiv \int_0^\infty dk\ k^c \exp[-(ak^d + bk^{-d})] = \frac{1}{d}\left(\frac{b}{a}\right)^{\frac{1+c}{2d}} K_{\frac{1+c}{d}}(2\sqrt{ab}) ,$$

$$J(a,b,c) \equiv \int_{-\infty}^\infty dk \exp[i(c - bk + \frac{1}{2}ak^2)] = \sqrt{\frac{\pi}{a}}\left[e^{-i\frac{2b^2-ac}{a}} + ie^{i\frac{2b^2-ac}{a}}\right] ,$$

$$K(a,b,c,d) = \int_{-\infty}^\infty dk \exp[i(-d + ck - \frac{1}{2}bk^2 + \frac{1}{3}ak^3)] =$$
$$\frac{2\pi}{a^{1/3}} e^{i\left[\frac{b^2}{2a}\left(c - \frac{b^2}{6a}\right) - d\right]} \mathrm{Ai}\left(\frac{1}{a^{1/3}}\left[c - \frac{b^2}{2a}\right]\right) ,$$

where $\mathrm{Ai}(z)$ is the *Airy Function*, defined by

$$\mathrm{Ai}(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}} K_{1/3}\left(\frac{2}{3}z^{3/2}\right) ,$$

where $K_\nu(z)$ is the modified *Bessel Function* (sometimes known as the *Basset Function*) of order ν:

$$K_\nu(z) = \int_0^\infty dt\ e^{-z\cosh t} \cosh \nu t .$$

Another useful expression for the Airy Function is

$$\mathrm{Ai}(z) = \frac{1}{3^{2/3}\Gamma(2/3)}\left(1 + (1)\frac{z^3}{3!} + (1\cdot 4)\frac{z^6}{6!} + (1\cdot 4\cdot 7)\frac{z^9}{9!} + \cdots\right) -$$
$$\frac{1}{3^{1/3}\Gamma(1/3)}\left(z + (2)\frac{z^4}{4!} + (2\cdot 5)\frac{z^7}{7!} + (2\cdot 5\cdot 8)\frac{z^{10}}{10!} + \cdots\right) \ .$$

The Basset Function has a similar looking expansion in terms of the difference of two infinite series. Both series converge for all values of z.

(D) Dirty tricks in mathematical physics: Viewed as *functions* of their arguments, a, b, c, d, it is possible in some cases to differentiate H, I, J and K with respect to these arguments to obtain more general integrals where powers of k multiply the various exponentials! Another more legitimate trick is to notice that the J and K integrals are Fourier Transforms, and so each comes equipped with an inverse transform, wherein the quantity on the right side of each equations sits within an integral (integrated db and dc, respectively) containing a factor of $\exp(ibk)$ and $\exp(-ick)$ respectively. The left side of each of these equations is (up to a factor of 2π that you have to get in the right place, numerator or denominator!) the integrands of J and K with the $\exp(-ibk)$ and $\exp(ick)$ factors omitted. Finally, because Fourier Transforms satisfy a convolution theorem, you can build additional even more complicated results by using the two Fourier Transform pairs provided by J and K.

(E) Even dirtier tricks: Go back now to part (B) and use the spherical and cylindrical coordinate transformations to derive even more interesting integrals knowing what the answer has to be from part (C)! Challenge your friends to do these really nasty looking integrals with Bessel functions with several beers as a wager.

Exercise 4: MAGNETIC HELICITY & GAUGE SYMMETRY

One often casually dismisses the ability to select an arbitrary gauge in electromagnetism based on the the premise that the electric and magnetic fields are unaffected. But is this always the case?

(A) The *magnetic helicity* $H_{\mathbf{B}}$ is defined to be the the net "linkage" of the magnetic field lines in a closed volume:

$$H_{\mathbf{B}}(t) \equiv \int d\mathbf{x}\ \mathbf{A}\cdot\mathbf{B} = \int d\mathbf{x}\ \mathbf{A}\cdot(\nabla\times\mathbf{A}) \ .$$

Determine what must be true about the volume and the magnetic field in order that the value of the magnetic helicity is unchanged by an arbitrary gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\chi \ .$$

(B) How does the potential ϕ have to change under this gauge transformation if $\chi = \chi(\mathbf{x})$? What if χ depends upon time, t—how does your answer change?

(C) What is the meaning of

$$H_{\mathbf{J}}(t) \equiv \int d\mathbf{x}\ \mathbf{B}\cdot(\nabla\times\mathbf{B}) \ ?$$

(D) Can you derive a conservation equation for the magnetic helicity density $dH_{\mathbf{B}}/dt$? (Hint: first prove the conservation law

$$\frac{\partial}{\partial t}\mathbf{A}\cdot\mathbf{B}+\nabla\cdot(\phi\mathbf{B}+\mathbf{E}\times\mathbf{A})=-2\mathbf{E}\cdot\mathbf{B}\ .$$

Is this gauge invariant?)

(E) A closely allied topological concept to helicity is the *linkage* of two curves in three dimensional Euclidean space. Let Γ_1 and Γ_2 be two closed and non-intersecting curves, then their linkage integral is

$$\mathcal{L}_{12}\equiv\frac{1}{4\pi}\oint\oint(d\mathbf{x}_1\times d\mathbf{x}_2)\cdot\frac{\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|^3}$$

where the integrations take place along the two curves Γ_1 and Γ_2. Build yourself two very simple closed non-interesecting curves and carry out the integral for the case in which the curves are linked and unlinked!

Exercise 5: FORCE-FREE ELECTROMAGNETIC FIELDS

If the Lorentz Force, or equivalently the coupling term,

$$\rho\mathbf{a}^{EM}=\delta\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}$$

vanishes, the electromagnetic fields exert no forces on the material. Electromagnetic waves on their own manage to do this because they have $\delta=0$ and $\mathbf{J}=0$, identically. And of course, $\mathbf{E}=\mathbf{B}=0$ succeeds but in a trivial fashion.

(A) Is it possible to arrange a static ($\partial/\partial t=0$) nontrivial solution of Maxwell's Equations for which $\delta\mathbf{E}$ and $\mathbf{J}\times\mathbf{B}$ are both identically zero?

(B) How about a static solution where

$$c\delta\mathbf{E}=-\mathbf{J}\times\mathbf{B}\ ?$$

(C) If you were successful in either (A) or (B), were you also successful in *completely* eliminating the influence of the electromagnetic field on the material by also ensuring that

$$\mathbf{J}\cdot\mathbf{E}=0\ ?$$

Exercise 6: YET MORE MURaM

Figures 7, 8 and 9 are taken from the MURaM simulation described in Scene 1.

(A) Compute the horizontal and temporal means of the Maxwell Equations and convince yourself that

$$\langle B_3\rangle=\text{constant}$$

whereas $\langle B_1\rangle$ and $\langle B_2\rangle$ are not similarly constrained. Figure 7 shows (red circles)

$$\langle B_\perp\rangle\equiv\left(\langle B_1\rangle^2+\langle B_2\rangle^2\right)^{1/2}\ ,$$

and (blue circles)

$$\langle|\mathbf{B}|^2\rangle^{1/2}\ .$$

The plot of $|\langle B_3 \rangle|$ lies off the bottom of this plot, consistent with the initial condition of the simulation that there was no net flux in the vertical (gravity aligned) direction and a weak horizontal *seed* field. Notice that the simulation makes a lot of magnetic field, but not very much magnetic flux. This is an essential feature of a small-scale (i.e. local) magnetic dynamo. See, for example, Weiss & Proctor [**WP 1**] cited in Scene 2.
(B) Figure 8 gives the horizontal and temporal average of the quantity

$$\langle \frac{|\mathbf{B}|^2}{8\pi p} \rangle \equiv \langle \frac{B_1^2 + B_2^2 + B_3^2}{8\pi p} \rangle \ .$$

What can you conclude about the importance of the electromagnetic forces in the upper and lower potions of the computational domain?
(C) Figure 9 gives the relative contributions of the various components (right side of equation) of the total (left side of equation) plotted in Figure 8. What does this plot indicate about magnetic buoyancy?

7. Further Reading

There is truly no shortage of monographs and textbooks on electromagnetic theory and its usual large-scale quasi-static incarnation as magnetohydrodynamics in highly-conducting fluids. This is another case of seek the author who is your muse and stick with their writings on the subject.

For an overall practical introduction that covers just about everything at an appropriate level and does so without a lot of excess verbiage, Jackson [**J 1**] really leaves very little to complain about. It's also in Gaussian units, which, for me is a plus, obviously.

The antithesis in a certain sense, of Jackson, is the incredibly deep and extremely elegant treatment of the same basic material by
⋆[**R 2**] F. Rohrlich, Classical Charged Particles. Foundations of Their Theory, (Reading, MA: Addison-Wesley Publishing Company; 1965), xiii+305.

Maxwell's Equations and electromagnetism have been with us since the late 19th Century. The modern approach to the subject as provided by Jackson and Rohrlich misses some of the more curious symmetry aspects that were more prevalent in ealier treatments. Two are well worth the effort to find and deal with the cumbersome notation that was used at the time. They are
[**B 3**] H. Bateman, The Mathematical Analysis of Electrical and Optical Wave Motion. On the Basis of Maxwell's Equations, (New York, NY: Dover Publications; 1955), vii+159,
[**S 4**] G.A. Schott, Electromagnetic Radiation. And the Mechanical Reactions Arising From It, (Cambridge, UK: Cambridge University Press; 1912), xxii+330.
Harry Bateman, it must be said, was one of the most accomplished applied mathematicians of his generation—his papers and monographs are an amazing display of cunning, insight and mathematical virtuosity. George Schott, was among other things, devising ingenious means to find distributions of electric charge in motion that would, as an assembly, not radiate electromagnetic waves in the far-field. That is, before quantum mechanics, he was trying to reconcile

that fact that electrons could "orbit" nuclei, without continuously radiating energy and spiralling inward. Although an individual charge in an orbit must emit radiation because of its acceleration, if one superposed a very large number of infinitesimal charges in a variety of orbits, one could arrange that the phases of the emitted waves cancelled the radiation in the far-field. A wonderfully intricate and as we now know, incorrect, solution to a pressing problem of the time.

The significance of the gauge invariance of Maxwell's Equations was perhaps not completely appreciated initially. Later, when we learned of the weak and strong nuclear forces, and especially when efforts were made to unify the weak and electromagnetic forces, the significance of gauge fields and gauge symmetries became more apparent. A very accessible, and unique, discussion of these ideas from a historical perspective can be had in
[**M 2**] K. Moriyasu, An Elementary Primer for Gauge Theory, (Singapore, SG: World Scientific Publishing; 1983), xi+177.

Turning to magnetohydrodynamics, there is again a vast and rapidly expanding literature. I offer five selections below which in my opinion stand out above the crowd. Gene Parker, Paul Roberts, Leon Mestel and Russell Kulsrud [**K 1**] in no small way shaped modern magnetohydrodynamics through their wide-ranging applications of the methods to a great many astrophysical and geophysical problems. It's worth studying the subject from the masters:
⋆[**P 4**] E.N. Parker, Cosmical Magnetic Fields. Their Origin and Activity, (Oxford, UK: Clarendon Press; 1979), xvii+841,
[**R 3**] P.H. Roberts, An Introduction to Magnetohydrodynamics, (New York, NY: American Elsevier Publishing Company; 1967), x+264,
[**M 3**] Leon Mestel, Stellar Magnetism, 2nd Edn, (Oxford, UK: Oxford University Press; 2012), xxi+715.

Of the more contemporary treatments of the subject, the following are particularly noteworthy for a variety of reasons as I will explain.
[**D 1**] P.A. Davidson, Turbulence in Rotating, Stratified, and Electrically Conducting Fluids, (Cambridge, UK: Cambridge University Press; 2013), xvii+681, as the title suggests, covers almost everything—including *turbulence*—except radiation. Peter Davidson writes well and he does a superb job of bringing a lot of diverse ideas and research together in a coherent treatment. Meanwhile, the pair of volumes
[**GS 1**] J.P. Goedbloed & Stefan Poedts, Principles of Magnetohydrodynamics. With Applications to Laboratory and Astrophysical Plasmas, (Cambridge, UK: Cambridge University Press; 2004), xvi+613,
[**GKP 1**] J.P. Goedbloed, Rony Keppens & Stefan Poedts, Advanced Magnetohydrodynamics. With Applications to Laboratory and Astrophysical Plasmas, (Cambridge, UK: Cambridge University Press; 2010), xvi+634,
are probably the two books to own if *you* are only going to own two books on magnetohydrodynamics. If it is not in these volumes, you don't need to know it! Hans Goedbloed is to laboratory magnetohydrodynamics as Parker, Roberts, Mestel and Kulsrud are to astrophysical MHD. These two volumes are about as

good as it gets. But, as you might guess by now, my own *personal* preference for the one book to own is a very curious and yet truly amazing compendium of mathematics applied to magnetofluids by one of the great under-appreciated and prolific geniuses of our time,
[**W 1**] Gary Webb, Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws, Lecture Notes in Physics 946, (Cham, CH: Springer; 2018), xiv+301.

Of the four integrals provided to you in the *exercise* on wavepackets, the first three are fairly elementary and can be found in a variety of tables. The third is related to Fresnel Integrals in optics. The fourth, however, is really a rare beast of a very different color. It can be found in
⋆[**BB 1**] M.V. Berry & N.L. Balazs, "Nonspreading wave packets", *American Journal of Physics*, **47**(3), 264-7, 1979.
It pays to have a few references handy to help you deal with all the special functions, like the Gamma Function, and the Airy Function, and Bessel Functions which are part of the repertoire of mathematical physics. For this *exercise* I used
⋆[**AS 1**] Milton Abramowitz & Irene A. Stegun, Handbook of Mathematical Funtions with Formulas, Graphs and Mathematical Tables, (Washington, DC: National Bureau of Standards; 1964), xiv+1046,
of which there are many later (and cheaper) Dover Publication editions. In fact, the entire book is now free and online via, https://dlmf.nist.gov/ under a different title.
Also useful is the reference book-as-car-manual approach of
⋆[**SO 1**] Jerome Spanier & Keith B. Oldham, An Atlas of Functions, (New York, NY: Hemisphere Publishing; 1987), ix+700.
Finally, my personal favorite is the more wordy but very comprehensive approach of
[**L 2**] N.N. Lebedev, Special Functions & Their Applications, (New York, NY: Dover Publications; 1972), xii+308.

Topological methods in fluid dynamics and electromagnetism are treated by
[**BF 1**] Mitchell A. Berger & George B. Field, "The topological properties of magnetic helicity", *Journal of Fluid Mechanics*, **147**, 138-48, 1984.
[**B 5**] Mitchell A. Berger, "Third-order link integrals", *Journal of Physics A: Mathematical and Theoretical*, **23**, 2787-93, 1990,
[**WB 1**] Andrew N. Wright & Mitchell A. Berger, "A physical description of magnetic helicity evolution in the presence of reconnection lines", *Journal of Plasma Physics*, **46**(1), 179-99, 1991,
[**B 6**] Mitchell A. Berger, "Introduction to magnetic helicity", *Plasma Physics and Controlled Fusion*, **41**, B167-75, 1999.
Indeed, the Wright & Berger paper draws attention to the interesting fact that reconnection and dissipation of magnetic fields converts (via the $\mathbf{J} \cdot \mathbf{E}$ term) magnetic energy into internal material energy but largely preserve magnetic helicity.

Although no one has stumbled across any yet, there seems to be no fundamental reason why there cannot be magnetic charges, called *monopoles*, in addition to electric charges. Since it is best to concentrate on things that *do* seem to exist rather than those that don't, I offer the following two articles in the spirit of light science-fiction reading:

[**GK 1**] Delbert Garrick & Raymond Kuselman, "Magnetic monopoles", *The Physics Teacher*, **9**(7), 366-9, 1971,

[**R 5**] Arttu Rajantie, "The search for magnetic monopoles", *Physics Today*, **69**(10), 40-6, 2016.

8. Appendix A: Spherical Geometry

To express Maxwell's Equations in spherical coordinates one only needs the relevant expressions for the divergence and curl of a vector field. These are

$$\nabla \cdot \mathbf{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ B_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}\ ,$$

$$(\nabla \times \mathbf{B})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ B_\phi\right) - \frac{1}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi}\ ,$$

$$(\nabla \times \mathbf{B})_\theta = \frac{1}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\phi\right)\ ,$$

$$(\nabla \times \mathbf{B})_\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) - \frac{1}{r}\frac{\partial B_r}{\partial\theta}\ .$$

9. Appendix B: Units, Dimensions and All That, Again...

Back in Scene 1 §2 we pointed out that the gravitational force per unit mass, i.e., the acceleration experienced by a parcel of fluid at point $\mathbf{x}$ exerted by a different element of fluid of volume $d\mathbf{x}'$ located at $\mathbf{x}'$ is

$$G\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}',t)d\mathbf{x}'$$

where G is Newton's Constant. The dimensional constant, G, requires that we measure mass in grams (or some multiple thereof, like kg, in the SI system of units).

Now let each parcel of fluid also carry an electric charge density $\delta(\mathbf{x},t)$ and $\delta(\mathbf{x}',t)$ respectively, then the electrostatic force per unit mass, i.e., the acceleration experienced by a parcel of fluid at the point $\mathbf{x}$ exerted by a different element of flud of volume $d\mathbf{x}'$ located at $\mathbf{x}'$ is experimentally found to be

$$K\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\delta(\mathbf{x}',t)d\mathbf{x}'\cdot -\frac{\delta(\mathbf{x},t)}{|\delta(\mathbf{x},t)|}\ .$$

The additional factor is ± 1 and accounts for the fact that electric charge can be positive and negative, with the like charges repelling and opposite charges attracting one another. The value and dimensions of the factor K depend likewise on what units we choose to measure the electric charge. Different choices for K lead to different systems of units.

The Gaussian-cgs-Kelvin system chooses K to be the dimensionless constant, 1. Charge is then measured in what are called electro-static-units, or e.s.u., as a common abbreviation, which must have dimensions of gm$^{1/2}$ cm$^{3/2}$ sec^{-1} in order that this last expression have units of acceleration, cm sec^{-2}. In these units the fundamental charge on an electron and proton is

$$e = 4.8032... \times 10^{-10} \text{ e.s.u. ,}$$

which then leads to the ubiquitous expression for the fine structure constant

$$\alpha \equiv \frac{2\pi e^2}{hc} = \frac{e^2}{\hbar c} = \frac{1}{137.03599...} .$$

This is the system we use in these notes, and which was generally preferred in physics for some time because the notation is fairly simple, if you don't mind carrying about some factors of 4π in your equations. For some people these 4π's proved to be unnerving and they picked a different numerical value for K, but one which was still dimensionless, in order to eliminate the two factors of 4π in Maxwell's Equations. These people are called "rationalists" and their units are "rationalized" because π, afterall, is *ir*rational.

Modern convention has headed off in the opposite direction, and has settled on a *dimensional* constant

$$K = \frac{1}{4\pi\varepsilon_0} ,$$

where the dimensional constant ε_0 is the permittivity of free space,

$$\varepsilon_0 = 8.8542... \times 10^{-12} \text{ farads m}^{-1} = \text{m}^{-3} \text{ kg}^{-1} \text{ sec}^4 \text{ ampere}^2$$

measured in farads—abbreviative F—per meter, and charges are now measured in coulombs—abbreviated C, with the fundamental charge now taking on the value

$$e = 1.602177... \times 10^{-19} \text{ C ,}$$

and the ampere—abbreviated A—is a measure of electric current, being one coulomb per second. Notice the propensity to create lots of named units in honor of people like Faraday, Coulomb and Ampère.

To make matters even more complicated, a second dimensional constant is introduced, the permeability of free space

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry m}^{-1} = \text{m kg sec}^{-2} \text{ A}^{-2}$$

(and another honoree, Henry) such that the speed of light, c, is exactly

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} .$$

Precisely why in a classical pre-Quantum-Electrodynamic universe, absolute vaccuum should possess a nonzero permittivity and permeability is not clear to me, although it is reassuring that they can be combined to give something that is meaningful for the vacuum—the propagation speed of disturbances like light and gravity.

With these SI conventions and honorific units in place, Maxwell's Equations read

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\delta \ , \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ ,$$

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \ .$$

Notice that the electric and magnetic fields now have different dimensions, volts and teslas, respectively (and two more honorees, Volta and Tesla). And this makes an SI version of RMHD rather cumbersome and a notational nightmare. Like many things in life, when it comes to the selection of K, you simply have to take your stand and then stand by it.

Finally, it is worth pointing out that one could think about taking G to be a dimensionless constant like 1 if we are willing to measure mass in the equivalent "gravitato-static-units", or g.s.u.'s, but this has not generated much enthusiasm.

10. Appendix C: RMHD's 58 Terms

The *energy density* of the electromagnetic fields is

$$\frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} \ ,$$

and the *energy flux* is the Poynting Vector

$$\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \ .$$

The *energy exchange term* with the material is

$$\dot{\mathcal{E}}^{M\to EM} - \dot{\mathcal{E}}^{EM\to M} = -\mathbf{J}\cdot\mathbf{E} \ .$$

The *momentum density* of the electromagnetic fields is

$$\frac{1}{c^2}\mathbf{S} \ ,$$

and the *momentum flux tensor* for the electromagnetic fields is

$$\mathtt{M} \ .$$

The *momentum exchange* term between the material and the electromagnetic fields is

$$\dot{\mathcal{P}}_i^{M\to EM} - \dot{\mathcal{P}}_i^{EM\to M} = -\delta E_i - \frac{1}{c}\epsilon_{ijk} J_j B_k \ .$$

Finally, the electromagnetic *field equations* are

$$\nabla \cdot \mathbf{E} = 4\pi\delta \ , \qquad c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ ,$$

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad c\nabla \times \mathbf{B} = 4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \ .$$