Act I. Scene 2: Euler & Navier-Stokes Equations

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September 16, 2018

1. Introduction

Euler (and others) set about the task of applying Newton's law of motion (accelerations are produced by forces) to an assembly of material (a gas or a fluid) in the continuum limit. We'll omit the derivation (but see Appendix B for some supplementary remarks) and simply quote the result, which is the requisite evolution equation for the Eulerian velocity

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = 0 \ .$$

This is known as *Euler's Equation.*

There are perhaps two surprises here. First, an additional scalar field, the gas (or fluid) pressure $p(\mathbf{x}, t)$ [dimensions: gm cm^{-1} sec^{-2}], appears. So we are still short one equation to close our system mathematically. Second, a nonlinear term is present

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla \cdot |\mathbf{u}|^2 + (\nabla \times \mathbf{u}) \times \mathbf{u} \ .$$

The curl of the velocity field is called the *vorticity* and is denoted by $\boldsymbol{\omega}(\mathbf{x}, t)$.

The pressure, like the density, is also a thermodynamic state variable. In the present context if the pressure is constant over some region then

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = 0 \ ,$$

in that region, and $\mathbf{u} = 0$ is a very reasonable solution. It is of course not the only possible solution. This simpler equation is known as *Burger's Equation* and it turns out to be anything but simple. It is, however, very well studied because it is the archetype of a nonlinear hyperbolic PDE. The vorticity satisfies the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = 0 \ ,$$

implying that the vortex lines are frozen into the fluid. Hence if the vorticity vanishes initially, it remains zero everywhere for all times. Then we can express $\mathbf{u} = \nabla \phi$ for some scalar potential $\phi(\mathbf{x}, t)$ and Burger's Equation can be simplified further to

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 = f(t)$$

where $f(t)$ is a arbitrary function of time.

In this sense the pressure characterizes the mechanical equilibrium of a system. When a system has relaxed to a state of constant pressure it has achieved mechanical equilibrium. Determining how, and if (in fact), it (ever) gets there is the object of radiation magnetohydrodynamics.

The $\mathbf{u}\cdot\nabla\mathbf{u}$ term is what makes fluid mechanics so difficult and interesting compared to quantum mechanics (say) which is fundamentally a linear theory at heart. The action of this term results in the familiar and ubiquitous presence of *turbulence*. To appreciate why this might be the case, imagine a sinusoidal velocity fluctuation at time $t = 0$,

$$u_1 = \sin kx_1$$

then this nonlinear term makes a contribution

$$-\frac{1}{2}k\sin 2kx_1$$

to $\partial u_1/\partial t$. So, unless the pressure gradient intervenes the flow develops structure at twice the wavenumber of the initial disturbance. This new wiggle will in turn produce additional wiggles at $3k$ and $4k$, and so on and so forth.

2. Solving the Euler Equations. Part 1

Mathematical progress on the four equations

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = 0\ .$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u} = 0$$

is possible, again, only if some benevolent individual provides us with some additional information to pin down one of our five unknown functions.

An interesting possibility (among all those available to us and the benevolent individual) is that the flow is *incompressible*, so

$$\nabla\cdot\mathbf{u} = 0\ .$$

When the ensuing flow speed is everywhere much smaller than the speed of sound, the incompressible assumption is not a bad approximation to the dominant (leading order) behavior of the dynamics. And it is for this reason that such an assumption is often useful to pursue. In general the above equations do not guarantee incompressibility unless the pressure is carefully adjusted at each time to ensure this outcome.

This careful adjustment leads to the additional necessary equation for the pressure

$$\nabla\cdot\frac{1}{\rho}\nabla p = -\nabla\cdot[\mathbf{u}\cdot\nabla\mathbf{u}]\ .$$

This is an elliptic second-order PDE, and strictly speaking this completes the formulation of the problem. In practice, of course, the solution of these five nonlinear equations is far from trivial.

Thanks to our gravitational detour in Scene 1 we know how to solve this equation,

$$\nabla p(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)}{4\pi}\nabla\int d\mathbf{x}'\frac{1}{|\mathbf{x}'-\mathbf{x}|}\nabla'\cdot[\mathbf{u}(\mathbf{x}',t)\cdot\nabla'\mathbf{u}(\mathbf{x}',t)]\ .$$

Of course, this is only half the pressure gradient because we alway get to add a term

$$\rho(\mathbf{x},t)\nabla\times\mathbf{A}(\mathbf{x},t)$$

to the right side of this equation. Now we need to determine $\mathbf{A}(\mathbf{x},t)$. The required equation is

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\mathbf{u})+\nabla\times\nabla\times\mathbf{A}=0$$

where

$$\boldsymbol{\omega}\equiv\nabla\times\mathbf{u}$$

is the vorticity. Although instructive, these manipulations do not get us any closer to actually solving this problem which ultimately requires a numerical treatment. This is true even in the case of modeling the dynamics of the Earth's oceans were it is acceptable to consider the density as being strictly constant independent of space and time.

3. Solving the Euler Equation. Part 2

Yet another possibility is that someone tells us how the pressure depends upon the density or vice-versa. As we shall see later, for dynamics that takes place at constant entropy, such a relation is a direct consequence of the laws of thermodynamics. In any case, to be definite, suppose

$$\rho(\mathbf{x},t)=\varrho(p)\ \text{ equivalently }\ p(\mathbf{x},t)=\Pi(\rho)\ ,$$

where to be clear, these functional forms for ϱ and Π are obtained by some other methods. Then we find

$$\frac{\partial\mathbf{u}}{\partial t}+\boldsymbol{\omega}\times\mathbf{u}+\nabla\left[\frac{1}{2}|\mathbf{u}|^2+\int^{p(\mathbf{x},t)}\frac{ds}{\varrho(s)}\right]=0\ .$$

$$\frac{\partial p}{\partial t}+\nabla\cdot p\mathbf{u}=0\ ,$$

with the very nice property

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\mathbf{u})=0\ .$$

If we are fortunate to have initial conditions in which the vorticity is zero, it will remain zero for all times. In this case we express the velocity as the gradient of a scalar function $\phi(\mathbf{x},t)$ and we immediately obtain the Bernoulli integral

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+\int^{p(\mathbf{x},t)}\frac{ds}{\varrho(s)}=f(t).$$

The lower limit of the integral is arbitrary and simply adds a constant to the unknown function $f(t)$. From this result we can formally solve for p and obtain a single nonlinear PDE for the potential ϕ.

4. Navier, Stokes and Some Other Folks

Euler overlooked, or perhaps better chose simply to ignore, an additional aspect of the behavior of assemblages of particles known as viscosity. Navier and Stokes worked out how to describe this tendency, especially for dense fluids like water, to oppose the build up of velocity shears. They replaced Euler's Equation with the *Navier-Stokes Equation*

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \frac{1}{\rho} \nabla \cdot \mathbb{\sigma} = 0 \ ,$$

or in component form

$$\frac{\partial u_i}{\partial t} + u_j \cdot \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{1}{\rho} \frac{\partial \sigma_{ji}}{\partial x_j} = 0 \ .$$

The viscous (sometimes referred to as the vicious) stress tensor $\mathbb{\sigma}(\mathbf{x}, t)$ [dimensions: gm cm^{-1} sec^{-2}] is defined by

$$\sigma_{ij} \equiv \mu \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big) + \lambda \delta_{ij} \nabla \cdot \mathbf{u} = 2\mu E_{ij} + \lambda \delta_{ij} \nabla \cdot \mathbf{u} \ .$$

Here, $\mu(\mathbf{x}, t)$ and $\lambda(\mathbf{x}, t)$ [dimensions: gm cm^{-1} sec^{-1}] are the coefficient of shear viscosity and dilatational coefficient of viscosity, respectively. The quantity $\mathbb{E}$ is the rate of strain tensor. For sensible (i.e., so-called Maxwellian) fluids,

$$\lambda = -\frac{2}{3} \mu \ ,$$

which ensures that $\sigma_{ii} = 0$. Clearly $\sigma_{ij} = \sigma_{ji}$. One must specify or impose a model for the microphysics in order to determine the unknown function $\mu(\mathbf{x}, t)$. The ratio μ/ρ has dimensions of cm^2 sec^{-1} and is sometimes denoted by $\nu(\mathbf{x}, t)$ and is the kinematic viscosity coefficient. If μ is independent of position, then the Navier-Stokes Equation takes the intuitive form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \nu \nabla^2 \mathbf{u} = 0 \ ,$$

indicating that velocity gradients nominally diffuse away on a characteristic time scale L^2/ν, provided the other two terms do not interfere.

The ratio

$$\mathrm{Re} = \frac{|\mathbf{u} \cdot \nabla \mathbf{u}|}{\nu |\nabla^2 \mathbf{u}|}$$

is a dimensionless quantity known as the *Reynolds Number*. And when it is large, as it is in most astrophysical applications, then the other two terms definitely interfere with this attempt at diffusion. Mathematically the problem becomes *singular*, because the coefficient in front of the highest spatial derivative is exceedingly small. For this term to come into play, there must exist very large gradients in (generally) isolated locations, often coincident with *boundary layers*. Euler's Equation is valid almost everywhere, except within the boundary layers where the full Navier-Stokes Equation is required. Where these boundary layers form, and how they evolve are challenging from a computational aspect.

The ratio

$$\mathrm{Ma} = \left(\frac{\rho |\mathbf{u} \cdot \nabla \mathbf{u}|}{\nabla p} \right)^{1/2} \approx \left(\frac{\rho |\mathbf{u}|^2}{p} \right)^{1/2}$$

is called the *Mach Number*—although sometimes that appellation is reserved for the final expression in this equation. Provided the spatial variations of the velocity and the presure are comparable the two expressions are essentially the same. The square of the Mach Number is the ratio of the fluid *ram pressure* to the thermal gas pressure. Therefore the Mach Number is also basically the ratio of the fluid velocity to the propagation speed of sound waves. Flows with low Mach Numbers require percentage-wise relatively small density and pressure fluctuations to balance the the ram pressure. For this reason, to leading order these flows can be treated as *incompressible*.

We may inquire whether there are other terms missing from Euler's Equation. As we demonstrated in Scene I, there is a gravitational acceleration $\mathbf{g}$ which accelerates a fluid parcel according to

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \frac{1}{\rho} \nabla \cdot \varpi = \mathbf{g} = -\nabla \Phi \equiv \mathbf{a}^G \ .$$

The three equivalent expressions on the right side of this equation are synonymous and can be employed interchangeably.

Taking an inspired cue from this equation we anticipate that both the radiation field ($\mathbf{a}^R$) and the largescale electromagnetic fields($\mathbf{a}^{EM}$) will exert forces on the material. Accounting for these additonal effects symbolically we may continue adding acceleration terms to the right side of this equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \frac{1}{\rho} \nabla \cdot \varpi = \mathbf{a}^G + \mathbf{a}^{EM} + \mathbf{a}^R$$

where EM stands for electromagnetic and R for radiation.

Sometimes it *might* prove advantageous to work in a noninertial frame of reference that rotates with angular velocity $\mathbf{\Omega}$ [dimensions: rad sec^{-1}]. This adds additional *non-inertial* terms

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p - \frac{1}{\rho} \nabla \cdot \varpi = \mathbf{g} + \mathbf{a}^{EM} + \mathbf{a}^R + \mathbf{a}^\Omega \ ,$$

where

$$\mathbf{a}^\Omega \equiv -2\mathbf{\Omega} \times \mathbf{u} + |\mathbf{\Omega}|^2 \mathbf{x} - \mathbf{\Omega}(\mathbf{\Omega} \cdot \mathbf{x}) - \frac{d\mathbf{\Omega}}{dt} \times \mathbf{x} \ .$$

However, a word of warning is in order here. One should really avoid working in non-inertial reference frames if at all possible. The reason is that it is extremely challenging (although not impossible) to figure out the correct forces exerted by the electromagnetic and radiation fields on the material in these non-inertial frames. The $\mathbf{a}^\Omega$ quoted here *only* takes account of the transformation of velocities and fixed vectors between these two frames, it does not know about $\mathbf{a}^{EM}$ or $\mathbf{a}^R$.

Although we already deduced that $\mathbf{g} = -\nabla\Phi$ and that

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{x}, t)$$

for a self-gravitating system, we will need to expend some additional efforts to determine $\mathbf{a}^{EM}$ and $\mathbf{a}^R$, and this is best done in an inertial reference frame. So for the remainder of these notes we will set $\boldsymbol{\Omega} = 0$ and ignore the coriolis and centrifugal acceleration effects.

5. Conservation of Momentum

If we multiply the generalized Euler Equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla\cdot\sigma = \mathbf{g} + \mathbf{a}^{EM} + \mathbf{a}^R \ ,$$

by ρ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{u} = 0$$

by $\mathbf{u}$ and add them together, we obtain

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla\cdot\Big(p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u}\Big) = \rho(\mathbf{g} + \mathbf{a}^{EM} + \mathbf{a}^R) \ ,$$

where $\mathbb{1}$ is the unit tensor

$$\nabla\cdot p\mathbb{1} = \nabla p \ .$$

The time derivative of the momentum flux density carried by the material plus the divergence of the momentum flux density is equal to the applied gravitational, electromagnetic and radiative forces.

If, and indeed *only if*, the fluid is self-gravitating, the fluid density follows directly in terms of the Laplacian of the gravitational potential,

$$\rho(\mathbf{x}, t) = \frac{1}{4\pi G}\nabla\cdot\nabla\Phi \ .$$

This equation is also true if some part of the potential $\Phi(\mathbf{x}, t)$ is supplied by material lying outside of the region of interest, because it contributes nothing to the Laplacian. If, however, the self-gravity of the material is *not* taken into account in the formulation, which is often the case in treating the envelopes of stars and planets, then the right side of this equation is in fact zero, and says absolutely nothing about the fluid density, period.

For a self-gravitating system, where this equation is meaningful, we can multiply by $\mathbf{g}$ to obtain

$$\rho\mathbf{g} = -\rho\nabla\Phi = -\frac{1}{4\pi G}(\nabla\Phi)\ \nabla\cdot\nabla\Phi$$

or in component form

$$\rho g_i = -\rho\frac{\partial\Phi}{\partial x_i} = -\frac{1}{4\pi G}\frac{\partial\Phi}{\partial x_i}\frac{\partial^2\Phi}{\partial x_j\partial x_j} = -\frac{1}{8\pi G}\frac{\partial}{\partial x_j}\Big(2\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j} - \delta_{ij}\frac{\partial\Phi}{\partial x_k}\frac{\partial\Phi}{\partial x_k}\Big) \equiv -\frac{\partial G_{ij}}{\partial x_j} \ .$$

The gravitational stress tensor $\mathbb{G}$ is symmetric:

$$\mathbb{G} \equiv \frac{1}{8\pi G}(2\mathbf{gg} - \mathbb{1}|\mathbf{g}|^2) \ .$$

For a *self-gravitating fluid*, the conservation of momentum then takes the form:

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla \cdot \Big(p\mathbb{1} - \varpi + \rho\mathbf{uu} + \mathbb{G}\Big) = \rho(\mathbf{a}^{EM} + \mathbf{a}^{R}) \ ,$$

where the gravitational force has been converted into the divergence of an additional component of the momentum flux density.

For an astrophysical system in which one does *not* explicitly solve for the gravitational potential from Poisson's Equation (in otherwords one supplies an externally generated $\Phi(\mathbf{x}, t)$ that is a solution of Laplace's Equation), one must be content to leave the equation as

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla \cdot \Big(p\mathbb{1} - \varpi + \rho\mathbf{uu}\Big) = \rho(\mathbf{g} + \mathbf{a}^{EM} + \mathbf{a}^{R}) \ .$$

6. Summary

We succeeded in obtaining three equations for $\mathbf{u}(\mathbf{x}, t)$ to complete the continuity equation, but at the cost of introducing (at least) one additional quantity $p(\mathbf{x}, t)$, the gas pressure. This means we are again (at least) one equation short of completing our system. For a self-gravitating fluid, we also need an additional function, $\Phi(\mathbf{x}, t)$, but, thankfully it comes with its own (Poisson) equation. Otherwise, we need to supply a gravitational acceleration based on some additional considerations which lie outsides of our radiating magnetofluid.

We pointed out how the dynamics requires additional pseudoforces in rotating non-inertial frame with an angular velocity $\mathbf{\Omega}(t)$ relative to a fixed inertial frame. This leads to the concepts of the coriolis force and (centripetal acceleration) the centrifugal force. We remind the reader that Maxwell's Equations and the equations of radiative transfer *must* be properly constructed by some means in the rotating non-inertial frame to obtain the correct interactions with the material!

We also encountered our first dissipative process, viscosity, which requires the specification of a transport parameter $\mu(\mathbf{x}, t)$ based on some knowledge of the microphysics. And, of course, we have left open the very real possiblility that the large scale electromagnetic fields and the radiation can exert forces on the fluid through the two place holder $\mathbf{a}^{EM} + \mathbf{a}^{R}$. In the subsequent chapters we set about determining evolution equations for p, $\mathbf{a}^{EM}$, and $\mathbf{a}^{R}$.

7. Exercises

Exercise 1: More MURaM

Use the information in Figures 4, 5, and 6, along with Figures 1, 2 and 3 from the previous Scene to estimate both the mean Reynolds Number and the mean Mach Number for the MURaM simulation described in Act I Scene 1 §4. For the Mach Number, you have all the information you need, but, as Figure 5 demonstrates very clearly, the mean-square of the fluid velocity,

$$\langle u_1^2 + u_2^2 + u_3^2 \rangle = \langle u_1 \rangle^2 + \langle u_2 \rangle^2 + \langle u_3 \rangle^2 + \langle {u'_1}^2 \rangle + \langle {u'_2}^2 \rangle + \langle {u'_3}^2 \rangle$$

is completely dominated by the last three terms in this equation. So the mean flow, $|\langle \mathbf{u} \rangle|$ and the random turbulent convective flows, $\langle |\mathbf{u}|^2 \rangle^{1/2}$, are characterized by very different Mach Numbers!

For the Reynolds Number you can assume a Maxwellian fluid, but you still need to know what the coefficient of shear viscosity is for the MURaM fluid. This is a microphysics transport parameter, which we will address in the last Scene of this Act. A very reasonable approximation for the solar photosphere is

$$\mu = 1.2 \times 10^{-16} \left(\frac{T}{1 \text{ deg K}} \right)^{5/2} \text{ gm cm}^{-1} \text{ sec}^{-1} \; .$$

You will again face the interesting issue that the Reynolds Numbers for the mean and turbulent flows are dissimilar. How does this affect your choice for the characteristic length scale $L \approx 1/|\nabla|$ needed in each case? An additional useful piece of information about MURaM which may be helpful in this regard is that it has 256 equally spaced grid points in the vertical x_3 direction (if you count the circles in each plot!) and each computational cell has the same extent in each of the three directions of 15.7 km.

8. Further Reading

I agonized over how little or how much to put in this Scene. With the continuity equations (Act I Scene 1) and the Euler-Navier-Stokes Equation in hand, we have the essential components of hydrodynamics or fluid mechanics. This material is abundantly available in many different forms and guises and you are perhaps best advised to find your favorite treatment—or hydrodynamic muse—and stick with it. Where all these sources vary, of course, is in the closure problem of relating the gas pressure $p(\mathbf{x}, t)$ back to the density $\rho(\mathbf{x}, t)$, or whether an effort is made to incorporate an energy equation (Act I Scene 5). Compressible and incompressible fluid dynamics take on very different aspects. The former is much richer in nonlinearity and the development of fine-scale structure, like shock fronts, over a wide range of scales.

With these caveats in mind, I offer the following selection of further reading.

An often overlooked treatment of hydrodynamics that is rich in unusual perspectives is

[**B 2**] Garrett Birkhoff, Hydrodynamics. A Study in Logic, Fact, and Similitude, (New York, NY: Dover Publications; 1955), xiii+186.

Birkhoff was an accomplished mathematician who made contributions in many diverse areas. Also worth looking at, for its elegance and powerful use of mathematics to solve real world problems is his more technical contribution

[**BZ 1**] Garrett Birkhoff & E.H. Zarantonello, Jets, Wakes and Cavities, (New York, NY: Academic Press; 1957), xii+353.

Almost everything, and perhaps even a little more, that you might ever wish to know about the Navier-Stokes Equation can be found in the obscure little gem of a book
[**DR 1**] Philip Drazin & Norman Riley, The Navier-Stokes Equation. A Classification of Flows and Exact Solutions, (Cambridge, UK: Cambridge University Press; 2006), x+196.

Geophysical fluid dynamics is characterized by the relative importance of rotation, and to a lesser degree, gravitational stratification, in the ensuing dynamics. Those interested in rapidly rotating stars, pulsars and gamma-ray bursters can learn much from the 20th Century endeavors of the geophysicists. Without doubt, the pinnacle of lucidity and clarity in this regard is
$\star$[**E 1**] Carl Eckert, Hydrodynamics of Oceans and Atmospheres, (New York, NY: Pergamon Press; 1960), xi+290,
but honorable mention should also be accorded to
$\star$[**P 2**] Joseph Pedlosky, Geophysical Fluid Dynamics, (New York, NY: Springer-Verlag; 1984), xii+624,
[**S 3**] Rick Salmon, Lectures on Geophysical Fluid Dynamics, (New York, NY: Oxford University Press; 1998), xiii+378.
And just for your amusement only, to see how advanced this topic is today—and also to learn about a zillion non-dimensional numbers you will never need to use—take a gander at
[**Z 2**] Radyadour Zeytounian, Asymptotic Modeling of Atmospheric Flows, (Berlin, DE: Springer-Verlag; 1990), xii+396.

We say so very little about *turbulence* in this *Opera* because frankly there is *so* very much to say about it and just *where* to begin? This omission should certainly *not* be construed to suggest that turbulence is unimportant. Also the arduous road from large-scale eddies to the dissipation length scale is quite different for dense fluids (like water) and tenuous—especially collisionless—plasmas. Likewise the formal theory and the laboratory and numerical experiments have progressed along such very different directions that it seems impossible to cite one or two references that cover the whole gamut and do not leave you with a skewed or biased perspective. The best advice I can give is to determine which of these very many facets of turbulence appeal to you, and then go hunting on the internet/library for materials and references. Good luck!

Of the very many expositions describing how to obtain the continuum fluid equations, and transport coefficients, from the kinetic Boltzmann Equation, I especially like
[**G 1**] Tamas I. Gombosi, Gaskinetic Theory, (Cambridge, UK: Cambridge University Press; 1994), xiv+297,
[**VK 1**] Walter G. Vincenti & Charles H. Kruger, Jr., Introduction to Physical Gas Dynamics, (Malabar, FL: Robert E. Krieger Publishing Company; 1965), xvii+538,
[**C 2**] Carlo Cercignani, Rarefied Gas Dynamics. From Basic Concepts to Actual Calculations, (Cambridge, UK: Cambridge University Press; 2000), xviii+320,

$\star$[**P 3**] Eugene N. Parker, <u>Conversations on Electric and Magnetic Fields in the Cosmos</u>, (Princeton, NJ: Princeton University Press; 2007), xiii+179,
in addition to the excellent treatements presented in Mihalas & Mihalas [**MM 1**] and Shu [**S 1**].

Aeroacoustics refers to the study of the emission of sound by unsteady fluid motions, which became rigorous, to some extent with Lighthill's pioneering work in the mid 20th Century. The following books provide a very comprehensive introduction to the methods and results:
[**G 2**] Marvin E. Goldstein, <u>Aeroacoustics</u>, (New York, NY: McGraw-Hill Book Company; 1976), xvii+293,
[**DF-W 1**] A.P. Dowling & J.E. Ffowcs-Williams, <u>Sound and Sources of Sound</u>, (Chichester, UK: Ellis Horwood Ltd; 1989), –+321,
[**CDF-WHL 1**] D.G. Crighton, A.P. Dowling, J.E. Ffowcs-Williams, M. Heckl & F.G. Leppington, <u>Modern Methods in Analytical Acoustics. Lecture Notes</u>, (London, UK: Springer-Verlag; 1992), xvii+738,
[**H 1**] M.S. Howe, <u>Theory of Vortex Sound</u>, (Cambridge, UK: Cambridge University Press; 2003), xiv+216.

Having figured out how sound is generated by unsteady flows, it then became of interest in certain computational settings to find ways of *suppressing* the sound generated by the flows in order to take larger time steps! This is sometimes facetiously described as "sound-proofing" the hydrodynamic equations, or the *anelastic approximation*. Two very nice references on what these methods are and why in particular they are needed are:
$\star$[**G 3**] Gary A. Glatzmaier, <u>Introduction to Modeling Convection in Planets and Stars. Magnetic Field, Density Stratification, and Rotation</u>, (Princeton, NJ: Princeton University Press; 2014), xiii+311,
[**WP 1**] N.O. Weiss & M.R.E. Proctor, <u>Magnetoconvection</u>, (Cambridge, UK: Cambridge University Press; 2014), xi+397.

Finally, because I can't resist, the following very diverse and intriguing compilation indicates some of the odd and interesting directions which fluid dynamics can take the researcher,
[**BMW 1**] G.K. Batchelor, H.K. Moffatt & M.G. Worster, eds., <u>Perspectives in Fluid Dynamics. A Collective Introduction to Current Research</u>, (Cambridge, UK: Cambridge University Press; 2003), xiii+631.

9. Appendix A: Spherical Geometry

The Euler Equation in spherical geometry is

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = g_r + a_r^{EM} + a_r^R \ ,$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{\cot\theta\ u_\phi^2 - u_r u_\theta}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = g_\theta + a_\theta^{EM} + a_\theta^R \ ,$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi(\cot\theta\ u_\theta + u_r)}{r} + \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} = g_\phi + a_\phi^{EM} + a_\phi^R \ .$$

For the Navier-Stokes Equation we need an expression for the divergence of the viscous stress tensor in spherical geometry. This is an extremely nasty expression, which can be found, among other places, spread across pages 87 and 88 of Mihalas & Mihalas [**MM 1**]. We won't bother to record it here given the high probability for typographical errors.

The equation for the conservation of momentum is

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla\cdot\Big(p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u}\Big) = \rho[\mathbf{g} + \mathbf{a}^{EM} + \mathbf{a}^{R}]\ .$$

If we denote the combination of quantities that appear in parenthesis as the stress-energy tensor $\mathbb{T}$, then the following equations are convenient to have in mind,

$$(\nabla\cdot\mathbb{T})_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ T_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi} - \frac{T_{\theta\theta}+T_{\phi\phi}}{r}\ ,$$

$$(\nabla\cdot\mathbb{T})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ T_{\theta\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} - \frac{\cot\theta\ T_{\phi\phi} - T_{r\theta}}{r}\ ,$$

$$(\nabla\cdot\mathbb{T})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{r\theta}\right) + \frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{2\cot\theta\ T_{\theta\phi} + T_{r\phi}}{r}\ .$$

10. Appendix B: 96 Tears and 30 Moments Later...

Harold Grad, more perhaps than anyone else, labored to place the fluid dynamic equations on a solid mathematical foundation. Curiously, or perhaps not, he never wrote a monograph on the subject. However, using Google, you can find pdf's of many of his papers and reports freely available from a variety of archives and sources. Find them and read them!

The starting point for all such endeavors is generally the collisional Boltzmann Equation for the evolution of the single-particle distribution function, $f^s(\mathbf{x},\mathbf{p},t)$ defined on a 6-dimensional phase space, for some species of particle, s. The density, fluid velocity, fluid pressure tensor, and so on are defined in terms of various *moments* of this distribution function integrated over the momentum part of the phase space, i.e.,

$$\rho(\mathbf{x},t) \equiv \sum_s m_s \int d\mathbf{p}\ f^s(\mathbf{x},\mathbf{p},t)\ .$$

Many species with different masses, m_s per particle, can co-exist. Because of the presence of a

$$\mathbf{p}\cdot\nabla f^s$$

term in Boltzmann's Equation, higher moments with respect to $\mathbf{p}$ invariably drive the time behavior of lower moments. For example, the fluid velocity enters into the continuity equation, and the pressure tensor enters into the Euler/Navier-Stokes Equation, and so forth.

Somewhere in this cascading process one has to simply discard higher order moments, or make a closure statement that some higher order moment can be related phenomenologically to a lower-order moment already in play. Of course we are already very familiar with such things when we say the pressure tensor, is actually a scalar times the unit tensor, and that scalar function can be described by an equation of state that depends only on the density and the internal energy per unit mass. This is technically referred to as a 5-moment closure scheme because the five independent moments we keep in play are the fluid density (1), the three components of the fluid velocity (3) and either the internal energy per unit mass or the pressure (1), which are related to one another by the equation of state.

Generally speaking, the more collision-dominated a fluid/gas is, the more likely a low-order closure scheme will work. Likewise, the closer the actual distribution function will be to the Maxwellian Distribution. Therefore a complementary approach, due originally to Chapman and Enskog, is to assume the distribution function is “close” in some sense to a Maxwellian, and use the Boltzmann Equation and the moments to determine the transport coefficients needed to describle viscosity, thermal conductivity, and electrical conductivity and so forth, based on the form of the prescribed departures from Maxwellian.

A great many books approach MHD, and thus RMHD, from this kinetic theory perspective. We have tried our best to avoid these complications, albeit with some unavoidable collateral damage.

11. Appendix C: Sound Proofing and Aeroacoustics

The momentum conservation equation for the material is

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \nabla\cdot\Big(p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u}\Big) = \rho\mathbf{a}^{\Sigma} \ ,$$

where the gravitational, electromagnetic and radiative accelerations have all been gathered into a single term. The mass conservation equation for the material is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u} = 0 \ .$$

Notice that the mass flux appears in the former and the latter, albeit acted upon by different differential operators. So the two equations can be combined to give

$$-\frac{\partial^2\rho}{\partial t^2} + \nabla^2\cdot\Big(p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u}\Big) = \nabla\cdot\rho\mathbf{a}^{\Sigma} \ ,$$

which is suggestive to say the least. Let’s regroup terms:

$$\frac{\partial^2\rho}{\partial t^2} - \nabla^2 p = -\nabla\cdot\left(\rho\mathbf{a}^{\Sigma} - \nabla\cdot\rho\mathbf{u}\mathbf{u}\right) \ ,$$

and drop the viscosity since the Reynolds Numbers are huge for most astrophysical flows of any consequence.

Now the left side of this equation is *almost* the wave equation for density fluctuations associated with linear acoustic waves, so let’s add and subtract the necessary term to make it so

$$\frac{\partial^2\rho}{\partial t^2} - a^2\nabla^2\rho = \nabla^2(p - a^2\rho) - \nabla\cdot\left(\rho\mathbf{a}^{\Sigma} - \nabla\cdot\rho\mathbf{u}\mathbf{u}\right) \ .$$

The quantity a [dimensions: cm sec^{-1}] is a constant "sound speed"

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

where the deriviative is taken at constant entropy (see Scene 5).

As it stands, this equation is still exact—no approximation has been made. Lighthill however made a very astute observation. If an astrophysical (or any) system can be partitioned into a compact region where all sorts of interesting dynamical processes are taking place albeit at low to moderate Mach Numbers, and a quiescent surroundings, basically at rest and possessing a more-or-less uniform sound speed, then the largest (dominant) terms for the dynamical medium live on the right side of this equation, while the dominant terms in the quiescent surroundings live on the left side of the equation! Each side must almost nearly balance in their respective separate regions.

The slight imbalance of the right side of this equation in the dynamic region therefore acts as a source term for acoustic waves that propagate away in the far (quiescent) surroundings. In this source region, the $\partial^2\rho/\partial t^2$ term is wholly negligible in comparison to the other terms and it may be discarded. Discarding this term, implies

$$\nabla \cdot \rho\mathbf{u} = 0 \ ,$$

which is the *anelastic approximation* for low Mach Number flows. It is equivalent to filtering sound waves out of the dynamical description of the astrophysical system. This can be very computationally useful when sound wave travel rapidly and have small amplitudes.

12. Appendix D: RMHD's 58 Terms

The *momentum density* for the material is

$$\rho\mathbf{u} \ ,$$

which is identical to the *mass flux*. The *momentum flux tensor* for the material is

$$\mathbb{T} = p\mathbb{1} - \sigma + \rho\mathbf{u}\mathbf{u} \ .$$

The *momentum density* for the gravitational field is zero.

The *momentum flux tensor* for the gravitational field of a *self-gravitating fluid* is

$$\mathbb{G} \ .$$

It vanishes if the gravitational field is supplied by external agents.

The *momentum exchange term* between the material and gravitational field is

$$\dot{\mathcal{P}}_i^{M\to G} - \dot{\mathcal{P}}_i^{G\to M} = -\rho g_i = \rho\frac{\partial\Phi}{\partial x_i} \ .$$

The gravitational *field equation* is

$$\nabla^2\Phi = 4\pi G\rho \ .$$