ACT I. SCENE 1: THE CONTINUITY EQUATION

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1. Introduction

At the outset, it might seem rather silly to devote an entire Scene to the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \; .$$

This equation expresses the well known fact that mass, as described in the continuum fluid approximation by the density $\rho(\mathbf{x}, t)$ [dimensions: gm cm⁻³], is neither created nor destroyed. The product $\rho \mathbf{u}$ is the mass flux, and $\mathbf{u}(\mathbf{x}, t)$ [dimensions: cm sec⁻¹] is the (Eulerian) fluid velocity.

To be definite, we shall assume (initially) that our radiation magnetohydrodyamics takes place on an inertial Euclidean three-space, $\mathbf{x} \equiv (x_1, x_2, x_3)$ [dimension: cm] and an independent one-dimensional (Euclidean) manifold called time, t [dimension: sec]. Hence the density and the three components of the fluid velocity are functions of fixed position and time on this four-dimensional Galilean space-time. We refer to them as the *Eulerian* fluid density and velocity. In a later Scene we will introduce the 10-parameter symmetry group of transformations that are necessary to make this a proper Galilean space-time in a precise sense.

An alternative expressions for the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \cdot \rho u_i = 0 ,$$

where a repeated index implies summation over all possible values (here, i = 1, 2, 3). We shall generally assume that ρ and the u_i are differentiable everywhere (although it may prove useful to relax this assumption when dealing with shock fronts in later applications). If the density is everywhere positive (or negative!) at some time t, it must be positive (or negative) for all times (past and future) because the time derivative of the density values when the density is zero and the density must pass through zero in order to change sign. This is a particularly nice property of this equation.

Mathematically speaking, the continuity equation is a first-order partial differential equation (PDE) for four *dependent* variables $\{\rho, u_1, u_2, u_3\}$ considered as functions of four *independent* variables $\{t, x_1, x_2, x_3\}$, which of course describe the underlying space-time. As such, there is not much else we can do at this point unless someone provides us with more information. Therefore, in Scene 2, we present an independent equation for **u**—the so-called Euler Equation or more generally the Navier-Stokes Equation—to secure some additional information. Indeed, if you wish to cut to the chase you can proceed there directly at this juncture.

2. Solving the Continuity Equation. Part 1

If, on the other hand, someone was kind enough to provide us with $\rho(\mathbf{x}, t)$, the continuity equation is sufficient to provide you with at least half of $\mathbf{u}(\mathbf{x}, t)$! To appreciate how this works, let us take a slight detour to develop some material we will need later.

At time t, consider an element of fluid situated at \mathbf{x} , then, according to Newton's theory of gravity, the gravitational force per unit mass (that is, the acceleration) exerted on that fluid element by a different element of fluid located at a position \mathbf{x}' , where the density is $\rho(\mathbf{x}', t)$ is

$$G\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}',t)d\mathbf{x}' \ .$$

Here $G = 6.6726... \times 10^{-8} \text{ cm}^3 \text{ gm}^{-1} \text{ sec}^{-2}$, is Newton's Constant, and $d\mathbf{x}' \equiv dx'_1 dx'_2 dx'_3$. Summing up the contributions from all fluid elements the *net* gravitational acceration experienced by the material located at \mathbf{x} is

$$\mathbf{g}(\mathbf{x},t) \equiv G \int d\mathbf{x}' \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \rho(\mathbf{x}',t) \ .$$

The dimensions of g are cm sec⁻² as one might hope. Next, observe that

$$abla rac{1}{|\mathbf{x}'-\mathbf{x}|} = rac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \; ,$$

 \mathbf{SO}

$$\mathbf{g}(\mathbf{x},t) = G \nabla \int d\mathbf{x}' \frac{1}{|\mathbf{x}' - \mathbf{x}|} \rho(\mathbf{x}',t) \equiv -\nabla \Phi(\mathbf{x},t) ,$$

where Φ [dimensions: cm² sec⁻²] is the gravitational potential. Remember that $\Phi(\mathbf{x}, t)$ is not unique, but can be adjusted by a gauge transformation

$$\Phi \rightarrow \Phi + \Phi_0$$

for an arbitrary constant Φ_0 with no impact on the dynamics.

Now we set about computing $\nabla \cdot \mathbf{g}$ using

$$abla \cdot rac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} = -rac{3}{|\mathbf{x}' - \mathbf{x}|^3} + 3 \; rac{|\mathbf{x}' - \mathbf{x}|^2}{|\mathbf{x}' - \mathbf{x}|^5} \; .$$

Notice that the right side of this equation is zero so long as $\mathbf{x} \neq \mathbf{x}'$. Therefore, in computing $\nabla \cdot \mathbf{g}$ from its expression as an integral over $\rho(\mathbf{x}', t)$ we may shrink the domain of integration to a small ball of radius ϵ centered on \mathbf{x} and let $\epsilon \to 0$,

$$\nabla \cdot \mathbf{g} = G \lim_{\epsilon \to 0} \int_{|\mathbf{x}' - \mathbf{x}| \le \epsilon} d\mathbf{x}' \rho(\mathbf{x}', t) \ \nabla \cdot \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \ .$$

Since ρ is differentiable everywhere, as $\epsilon \to 0$ we may replace $\rho(\mathbf{x}', t) \to \rho(\mathbf{x}, t)$ and remove it from the integrand. It remains to evaluate

$$\lim_{\epsilon \to 0} \int_{|\mathbf{x}' - \mathbf{x}| \le \epsilon} d\mathbf{x}' \nabla \cdot \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} = -\lim_{\epsilon \to 0} \int_{|\mathbf{x}' - \mathbf{x}| \le \epsilon} d\mathbf{x}' \nabla' \cdot \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3}$$

Gauss's law allows us to convert the volume integral to a surface integral:

$$\lim_{\epsilon \to 0} \int_{|\mathbf{x}'-\mathbf{x}| \le \epsilon} d\mathbf{x}' \nabla \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} = -\lim_{\epsilon \to 0} \oint_{|\mathbf{x}'-\mathbf{x}|=\epsilon} d\mathbf{S}' \cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} = -4\pi \ .$$

Thus,

$$\nabla \cdot \mathbf{g} = -4\pi G \rho(\mathbf{x}, t) \; ,$$

or equivalently

$$\nabla^2 \Phi = 4\pi G \rho$$

This is called Poisson's Equation for the gravitational potential consistent with Newton's law of gravity. It has the rather undesirable property that the gravitational potential reacts instantaneously everywhere in space to any change in the distribution of matter. It is this aspect of Poisson's Equation that Einstein's General Theory of Relativity corrects by the introduction of gravitational waves, which, have only recently finally been detected.

Returning to the continuity equation, we see that this is just the result we need to solve

$$abla \cdot
ho \mathbf{u} = -\frac{\partial
ho}{\partial t}$$

since we know how to construct **g** from ρ via a similar looking equation. We can immediately write down the desired result:

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{4\pi\rho(\mathbf{x},t)} \nabla \int d\mathbf{x}' \frac{1}{|\mathbf{x}'-\mathbf{x}|} \frac{\partial\rho(\mathbf{x}',t)}{\partial t}$$

This, however, turns out to be only half (or better, "part") of the answer! The reason is that we are free to add any *additional* velocity of the form:

$$\mathbf{v}(\mathbf{x},t) = \frac{1}{\rho(\mathbf{x},t)} \nabla \times \mathbf{A}(\mathbf{x},t)$$

where $\mathbf{A}(\mathbf{x}, t)$ is an *arbitrary* vector function of space-time, to \mathbf{u} , because we will still satisfy $\nabla \cdot \rho \mathbf{v} = 0$.

If it is also the case that $\nabla \times \rho \mathbf{u} = 0$, then we can conclude that $\nabla \times \rho \mathbf{v} = 0$. But this is still not sufficient to determine the unknown half of \mathbf{u} , because \mathbf{A} is now only restricted to nontrivial solutions of $\nabla \times (\nabla \times \mathbf{A}) = 0$. Equivalently we can still add to \mathbf{u} a velocity

$$\mathbf{w}(\mathbf{x},t) = \frac{1}{\rho(\mathbf{x},t)} \nabla \phi(\mathbf{x},t) ,$$

where ϕ is a solution of Laplace's Equation, $\nabla^2 \phi = 0$. So we are still able to recover only part of the velocity field from complete knowledge of the density! Even more information is obviously needed to construct the entire velocity field.

3. Solving the Continuity Equation. Part 2

Surprisingly (or perhaps not), if someone instead provides you with complete knowledge of $\mathbf{u}(\mathbf{x}, t)$ you may determine $\rho(\mathbf{x}, t)$ everywhere! However, the method to solve this problem is entirely different, and is (unfortunately) much more complicated (or powerful, take your choice).

The continuity equation is now a *linear* homogeneous PDE (of first order) for a scalar function ρ with specified coefficients:

$$\frac{\partial \rho}{\partial t} + u_i(\mathbf{x}, t) \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i(\mathbf{x}, t)}{\partial x_i} = 0 ,$$

in the four independent variables $\{t, x_1, x_2, x_3\}$. The method of solution, due to Lagrange, involves finding the characteristics, or integrals, of the partial differential equation. In other words, we seek a function $\Psi(t, \mathbf{x}, \rho)$, from which we can implicitly determine ρ from knowledge of a initial condition, $\rho_0(\mathbf{x}) = \rho(\mathbf{x}, t = 0)$, say. Then, as the fluid evolves dynamically, the relation $\Psi(t, \mathbf{x}, \rho) = \Psi(0, \mathbf{x}, \rho_0(\mathbf{x})) \equiv \psi_0(\mathbf{x})$, holds. Thus, we obtain an implicit solution for $\rho(\mathbf{x}, t)$ if we can find Ψ . Lagrange's method, as well as its extension to more general situations is discussed in Appendix E at the end of this Scene.

To proceed, we need to determine an equation for Ψ , which is:

$$\frac{\partial \Psi}{\partial t} + u_i \frac{\partial \Psi}{\partial x_i} - \rho \frac{\partial u_i}{\partial x_i} \frac{\partial \Psi}{\partial \rho} = 0$$

This follows by noting that on the surface $\Psi = \text{constant}$,

$$\frac{d\Psi}{dt} = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial\rho}\frac{\partial\rho}{\partial t} = 0 , \text{ at fixed } \mathbf{x} ,$$
$$\frac{d\Psi}{dx_i} = \frac{\partial\Psi}{\partial x_i} + \frac{\partial\Psi}{\partial\rho}\frac{\partial\rho}{\partial x_i} = 0 \text{ at fixed } t, x_j, j \neq i$$

Adding the first equation to u_i contracted with the second equation leads to the desired result. So we have another linear PDE now with *five* independent variables and with specified coefficient for the scalar function Ψ . It appears we might be headed in the wrong direction. But Lagrange is not to be doubted.

The *characteristics* of the PDE for Ψ are determined by the equations

$$\frac{dt}{1} = \frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3} = -\frac{d\rho}{\rho(\partial u_i/\partial x_i)} ,$$

which are in turn equivalent to the system of equations:

$$\frac{dx_i}{dt} = u_i(\mathbf{x}, t)$$
$$\frac{1}{\rho} \frac{d\rho}{dt} = -\frac{\partial u_i}{\partial x_i} \equiv -\Delta(\mathbf{x}, t)$$

For extremely simple $\mathbf{u}(\mathbf{x}, t)$ it may be possible to solve this set of equations analytically. For instance, if t does not appear explicitly, as in a steady flow, the system is said to be autonomous. In any case, with some luck, the idea is to determine four integrals of the motion, call them $\Omega_{\alpha}(t, \mathbf{x}, \rho)$ for $\alpha = 1, 2, 3, 4$ from these equations. And the general solution is then $\Psi = \omega(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$, where ω is an arbitrary function of the four integrals. But this rarely occurs in practice. So, instead we must take a different approach. Here, we move on to plan B.

As the right sides of all 4 equations are known, we may formally (or numerically, take your pick) integrate this system forward in time from an initial condition where a parcel of fluid has density $\rho_0(\mathbf{x}')$ at position \mathbf{x}' at time t = 0 to some final time t:

$$x_i = X_i(\mathbf{x}', t)$$

where

$$X_i(\mathbf{x}', t=0) = x'_i \; .$$

Knowledge of these three functions $\mathbf{X} = (X_i, X_2, X_3)$, of the initial position of a parcel of fluid $\mathbf{x}' = (x'_1, x'_2, x'_3)$, and the elapsed time t constitute a complete solution of our problem. Nothing more can be asked for. I've adopted the somewhat pedantic notation of writing

$$\mathbf{x} = \mathbf{X}(\mathbf{x}', t)$$

to carefully distinguish between an initial \mathbf{x}' and a subsequent \mathbf{x} location in Euclidean three-space. The vector function $\mathbf{X}(\mathbf{x}', t)$ is best regarded as a *mapping* from the initial position of a parcel of fluid \mathbf{x}' at time t = 0 to some subsequent position \mathbf{x} at time t generated by the Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$. In otherwords, \mathbf{X} is a continuous sequence of mappings from the Euclidean three-space to itself, with the time t serving the role of a continuous evolution parameter.

The Jacobian of the mapping

$$J_{ij}(\mathbf{x}',t) \equiv \frac{\partial X_i}{\partial x'_j} \; .$$

is particularly important. Strictly speaking, the Jacobian is a function of the parameter t and the initial location of the parcel of fluid \mathbf{x}' . However, it can also be regarded as a function of the position at time t of the parcel of fluid because of the mapping \mathbf{X} . Clearly, for all of this to hang together, we require the mappings to be one-to-one at every time t. In otherwords, $\mathbf{X}(\mathbf{x}', t) = \mathbf{X}(\mathbf{y}', t)$ if and only if $\mathbf{x}' = \mathbf{y}'$.

Equally important is the inverse mapping $\mathbf{X}'(\mathbf{x}, t)$ which provides the initial position \mathbf{x}' at t = 0 of a parcel of fluid which arrives at position \mathbf{x} at time t. Because the original mapping is one-to-one, we are guaranteed that the inverse exists and is also one-to-one! Its Jacobian is

$$J_{ij}'(\mathbf{x},t) \equiv \frac{\partial X_i'}{\partial x_j}$$

Again, this can also be regarded as a function of the initial position \mathbf{x}' because the inverse mappings \mathbf{X}' are also one-to-one. In terms of these forward (in time) and backward mappings, the Eulerian velocity is

$$u_i(\mathbf{x},t) = \frac{\partial}{\partial t} X_i(\mathbf{x}',t)$$
 evaluated at $\mathbf{x}' = \mathbf{X}'(\mathbf{x},t)$

Note that this requires knowledge of *both* the forward and backward maps. The expression

$$u_i(\mathbf{x}',t) = \frac{\partial}{\partial t} X_i(\mathbf{x}',t) ,$$

has exactly the same value, but it is attached to the point \mathbf{x}' , even though the parcel of fluid is now at the point \mathbf{x} . So it means a different thing. It is the Lagrangean velocity. The former is the Eulerian velocity. Again, they have precisely the same value but are attached to two different points in the Euclidean three-space at any given time (except t = 0), and so they are "different."

Now the essential critical property of the two Jacobians is the following:

$$J_{ij}J'_{jk} = \frac{\partial X_i}{\partial x'_j}\frac{\partial X'_j}{\partial x_k} = \delta_{ik} ,$$

provided we evaluate the two Jacobian's at the pair of positions \mathbf{x} and \mathbf{x}' which are related by the forward and backward mappings at the specified time t. In particular, notice that for t = 0, $\mathbf{x} = \mathbf{x}'$, $J_{ij} = \delta_{ij}$ and $J'_{jk} = \delta_{jk}$ so this property starts off being true at t = 0! And it remains so throughout the evolution.

We now have enough information to determine the density. The J_{ij} can also be regarded as the components of a 3x3 matrix, and provided we use the coordinates **x** and **x'** related by the forward and backward mappings at time t, J'_{jk} are therefore the components of the inverse of this 3x3 matrix. We now make use of standard results for the matrix inverse from elementary matrix theory to write

$$J_{jk}' = \frac{1}{2J} \epsilon_{klm} \epsilon_{jpq} J_{lp} J_{mq} ,$$

where the determinant of the the 3x3 matrix is

$$J \equiv \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} J_{il} J_{jm} J_{kn} \; .$$

The quantity ϵ_{ijk} is the Levi-Civita tensor and is the more complicated, and interesting, sibling of the Kronecker delta δ_{ij} . For Kronecker's delta,

$$\delta_{ij} = 0$$

if $i \neq j$, otherwise, its value is 1. Thus $\delta_{ii} = 3$ in our three-dimensional Euclidean space. For the Levi-Civita tensor,

$$\epsilon_{ijk} = 0$$

if any two indices are equal, irrespective of the value of the third, otherwise its value is 1 if $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$, i.e.,

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$
.

Exchanging any two indices flips the sign, so

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$$
.

The cross product of two vectors in Euclidean three-space makes use of the Levi-Civita tensor:

$$\mathbf{A} = \mathbf{B} \times \mathbf{C} \quad \Rightarrow \quad A_i = \epsilon_{ijk} B_j C_k \; .$$

A very useful identity is

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il} \ \delta_{jm} - \delta_{im}\delta_{jl} \ ,$$

which is simply a very complicated way of saying

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \; .$$

So, to complete the story, let's compute dJ/dt. Using the chain rule, we have

$$\frac{dJ}{dt} = \frac{\partial J}{\partial J_{ij}} \frac{dJ_{ij}}{dt}$$

The first factor can be computed directly from the definition of J as the product of three factors of $J_{il}J_{jm}J_{kn}$ and gives

$$\frac{dJ}{dt} = JJ'_{ji}\frac{dJ_{ij}}{dt} = JJ'_{ji}\frac{\partial^2 X_i}{\partial t\partial x'_j} = J\frac{\partial X'_j}{\partial x_i}\frac{\partial}{\partial x'_j}\frac{\partial X_i}{\partial t} = J\nabla \cdot \mathbf{u} = -\frac{J}{\rho}\frac{d\rho}{dt} \ .$$

Therefore,

$$\frac{d}{dt}J\rho = 0 \quad \Rightarrow \quad J(\mathbf{x}',t)\rho(\mathbf{x}',t) = \rho_0(\mathbf{x}') \; .$$

This deceptively simple statement implies that the density of a parcel of fluid, which was originally $\rho_0(\mathbf{x}')$ at time t = 0 when it was located at position \mathbf{x}' , becomes

$$\rho(\mathbf{x}',t) = \frac{\rho_0(\mathbf{x}')}{J(\mathbf{x}',t)} = \frac{\rho(\mathbf{x}',0)}{J(\mathbf{x}',t)}$$

at a later time t when that parcel of fluid is at $\mathbf{x} = \mathbf{X}(\mathbf{x}', t)$. The ratio of the densities for the parcel of fluid is proportional to the determinant of the Jacobian. This is a Langrangean statement because even though the fluid parcel is at \mathbf{x} at this time t, its actual density is attached back to the point \mathbf{x}' from whence it came.

To obtain the Eulerian density, we have to reattach this density to the actual location of the fluid parcel, or, in otherwords,

$$\rho(\mathbf{x},t) = \frac{\rho_0(\mathbf{x}')}{J(\mathbf{x}',t)} \text{ evaluated at } \mathbf{x}' = \mathbf{X}'(\mathbf{x},t) ,$$

is the desired solution to

$$\frac{\partial \rho}{\partial t} + u_i(\mathbf{x}, t) \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i(\mathbf{x}, t)}{\partial x_i} = 0 ,$$

where $\rho(\mathbf{x}, 0)$ is specified by the function $\rho_0(\mathbf{x})$.

Notice, of course, that without knowledge of \mathbf{X} and its partial derivatives $\partial X_i / \partial x'_j$ this is only a formal solution because, of course, we haven't actually solved anything. And the only way we get this information is going back to the system of four coupled ODEs (in time) and integrating them forward somehow from the initial condition. Our ability to do this hinges upon knowing the Eulerian velocity field everywhere at all times $t \geq 0$.

The quantity

$$\frac{\rho_0(\mathbf{x}')}{J(\mathbf{x}',t)}$$

when regarded simply as a function of the initial position of a parcel of fluid, \mathbf{x}' , and the elapsed time is called the *Lagrangean* fluid density. And by the same token,

 $\frac{\partial \mathbf{X}}{\partial t}$

regarded simply as a function of the initial position of a parcel of fluid, \mathbf{x}' , and the elapsed time is the *Lagrangean* fluid velocity. In fact, it is possible to treat the dynamics of a radiating magnetofluid entirely from a Lagrangean perspective, where a fluid element is characterized by its initial location and the elapsed time. Where a parcel of fluid ends up, and its physical attributes, at a certain elapsed time when it ends up there (like its density, pressure and magnetic field, for example) are completely determined by the mapping \mathbf{X} and its inverse. This is called the Cauchy solution. And as promised, it is powerful but not easy to arrive at. This is to be contrasted with the Eulerian approach where we wish to determine how various quantities evolve over a fixed Galilean space-time. Where things came from or where they are going don't enter the discussion.

Here, we determined the both the forward and backward mappings from complete knowledge of the Eulerian fluid velocity via

$$\frac{dx_i}{dt} = u_i(\mathbf{x}, t) \; ,$$

thanks to Lagrange. We ended up going down this road because the Eulerian velocity field was provided to us a priori. In practice, of course, no one just provides you with a velocity field. You have to figure it out. And anyway, if they had the Eulerian velocity field, then they also knew the density and everything else about the dynamics, and were not being particularly forthcoming in witholding this additional information from us. Live and learn.

4. An Application: MURaM

The continuity equation can be employed for a number of interesting applications. Here is one. Consider a very complicated numerical simulation of compressible, turbulent magnetoconvection complete with radiative transfer. To be definite, let's suppose we carry out the computation in a rectangular domain that is periodic in the two horizontal directions (say x_1 and x_2) that are perpendicular to gravity, with periodicity L [dimensions: cm]. The box has a depth H in the vertical (x_3) direction and the calculation is carried out for an elapsed time T [dimensions: sec].

The mean stratification achieved by the simulation over its duration is an interesting quantity. For example, the mean density stratification is

$$\langle \rho \rangle \equiv \frac{1}{L^2T} \int_0^T dt \int_0^L dx_1 \int_0^L dx_2 \ \rho(\mathbf{x},t) \ ,$$

which can only be a function of x_3 . Applying the $\langle \cdot \rangle$ operator directly to the continuity equation gives

$$\frac{d}{dx_3} \langle \rho u_3 \rangle = \frac{1}{L^2 T} \int_0^L dx_1 \int_0^L dx_2 \left[\rho(\mathbf{x}, 0) - \rho(\mathbf{x}, t) \right] \,,$$

because of the periodic boundary conditions in x_1 and x_2 .

Now, if the simulation is run for a sufficiently long time, and if it is reasonably well-behaved in the sense that the fluid density remains sensibly bounded, the term on the right side of this equation becomes increasingly negligible as $T \to \infty$ and so we conclude that

$$\langle \rho u_3 \rangle \rightarrow \text{constant}$$

which may, or may not, be zero depending upon the nature of the upper and lower boundary conditions in the long time limit.

But we can do a little better than this. Writing

$$\rho(\mathbf{x},t) = \langle \rho \rangle(x_3) + \rho'(\mathbf{x},t)$$

and a similar expression for $u_3(\mathbf{x}, t)$, we can conclude that

$$\langle \rho \rangle \langle u_3 \rangle + \langle \rho' u_3' \rangle = \langle \rho u_3 \rangle = \text{constant.}$$

So if we know the mean vertical velocity, the mean density, and the mean vertical mass flux through the computation domain (which again may be zero), we can determine the correlation of the density and vertical velocity flucutations about the mean (which need not be small in any sense of the word) without needing to compute it from the simulation! If the mass flux is zero and the mean vertical velocity vanishes, there can be no net correlation between the density and vertical velocity fluctuations. Conversely if there are correlations between the density and vertical velocity fluctuations (i.e., on average cool dense material settles and hot rarefied fluid rises) then there must be a mean vertical velocity if there is no mass flux through thel boundaries of the box! Conservation laws are useful.

Figures 1, 2 and 3 are based on just such a numerical simulation using the MURaM (Max-Planck-Institute for Aeronomy/University of Chicago Radiation Magneto-hydrodynamics) Code. This particular simulation lasts for approximately T = 30 min, and has H = 4 Mm and L = 6 Mm. It describes a typical patch of the quiet Sun from the bottom of the photosphere to a depth of about 2.5 Mm into the solar convection zone. We'll have more to say about, and show further illustrations from, this simulation throughout this Act and the next.

Using the formulae above, in Figure 1 we plot the average mass flux parallel $\langle \rho u_3 \rangle$, and perpendicular $[\langle \rho u_1 \rangle^2 + \langle \rho u_2 \rangle^2]^{1/2}$ to gravity as well as the product $\langle \rho \rangle \langle u_3 \rangle$. Consistent with our expectations, $\langle \rho u_3 \rangle$ is essentially a constant, and that constant is close to zero (the numerical errors are at worst 10^{-4} gm cm⁻² sec⁻¹).

This implies that to numerical round off errors

$$\langle \rho' u_3' \rangle = -\langle \rho \rangle \langle u_3 \rangle.$$

At $x_3 = 2.3376$ Mm, which basically marks the location of the solar surface in this simulation, this quantity changes sign. Since the mean density is positive definite (see Figure 2), it follows that there is an average upflow below the solar surface that turns into a downflow above the solar surface (see Figure 3). It also implies that below the solar surface, excess density correlates with downflows and density depletions with upflows. The correlations are opposite above the solar surface.

This picture is consistent with the nature of vigorous compessible convection. Cool dense material descends rapidly in an isolated network of narrow intergranular lanes which surround large gradual upwellings of hotter, and more rarefied, material. These more gentle upflows—called granules—cover 70-90% of the simulation (and solar) surface area, and they dominate an unweighted spatial average, producing the mean upflow in Figure 3. In the stably-stratified atmosphere overlying the surface, gradual subsidence of light material occupies more space than the concentrated, outflow jets—called spicules—of upward moving hot dense material. This change in flow topology is responsible for the sign change in the mean velocity and the mass flux correlation in Figures 1 and 3.

Another sign change occurs near the very top of the simulation, around $x_3 = 3.8$ Mm. This however, is a consequence of the upper boundary conditions which attempt to let material, what little there is at these altitudes, exit the simulation through a semi-permeable outflow boundary. As the mean density here is a few times 10^{-11} gm cm⁻³, this artifice has little impact on the dynamics and allows waves and radiation to exit the simulation.

5. Summary

We've managed to fill 10 pages with text and formulas concerning an equation which most authors mention in one line and then move on to more interesting things. But, hopefully, we have managed to convey some appreciation for the power of this equation and some of the subtleties associated with attempting to solve it under a variety of conditions. Along the way, we have had the chance to develop some useful mathematical tools from potential theory, the solution of partial differential equations, matrix theory, and the Lagrangean view of fluid dynamics. These things will all prove useful in subsequent Scenes and Acts, and hopefully later in your academic careers.

In any event, the bottom line is that the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \ ,$$

is one piece of the bigger radiation magnetohydrodynamic puzzle as it is one equation containing four unknowns. Therefore we will need at least three additional equations to arrive at a well-defined problem. In fact we will end up with a lot more than that. But we now proceed to Euler's Equation.

We also introduced, in passing, Newton's theory of gravitation, for the scalar gravitational potential $\Phi(\mathbf{x}, t)$,

$$\nabla^2 \Phi = 4\pi G \rho$$
.

Some authors prefer to use $-\Phi$ rather than Φ , in which case this equation picks up a "minus" sign, as does the definition of the gravitational acceleration, which for *our* sign convention reads:

$$\mathbf{g} = -\nabla \Phi$$
 .

Make the effort to keep the sign convention consistent in your work.

Philosophically, Newton's theory of gravitation suffers from the infinite speed of action-at-a-distance implicit in the Poisson Equation. This in turn will make it impossible to store any momentum in the gravitational field, and the energy stored in the gravitational field will have the unsatisfactory requirement that matter (via ρ) must be present for any storage to be realized. These issues, and other concerns, were paramount in Einstein's formulation of a causal theory of gravitation replete with gravitational waves that can store and transport energy and momentum in the absence of any matter. His General Theory of Relativity has three particularly novel aspects in that it is (*i*) manifestly a non-linear theory, (*ii*) it is a tensor theory, so there is not one scalar potential Φ but a whole menagierie of "potentials" that are assembled as the components of a tensor, and (*iii*) it is intimately tied to the space-time upon which it operates.

When fluid motions are slow—meaning speeds much less than the speed of light—and gravity is weak—meaning that the radius of curvature of space-time is very large compared to any spatial scale of interest, then a limiting expansion of General Relativity in inverse powers of the speed of light is possible. This expansion, which can be truncated in many different fashions depending upon the application in mind, is referred to as the *Post-Newtonian Approximation* to the General Theory of Relativity. The absolute zeroth-order limiting case, $c \to \infty$, and a flat Euclidean space-time, is just Poisson's Equation. To do better than this requires a significant investment, which for the present we shall choose to avoid.

6. Exercises

Exercise 1: POISSON'S EQUATION

The linear second-order PDE for $\psi(\mathbf{x})$ in an *n*-dimensional Euclidean space,

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \dots + \frac{\partial^2 \psi}{\partial x_n^2} = -f(\mathbf{x}) ,$$

for n = 1, 2, 3, ... is known as the *Poisson Equation*. For Newtonian gravity, we have the n = 3 version of this equation with the association $\Phi(\mathbf{x}) \to \psi(\mathbf{x})$, and

 $f(\mathbf{x}) \rightarrow -4\pi G \rho(\mathbf{x})$. Assuming that $f(\mathbf{x})$ has compact support (i.e., it vanishes everywhere outside a ball of finite radius), the *general* solution of this equation is

$$\psi(\mathbf{x}) = \int d\mathbf{x}' \Psi_n(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') ,$$

where, $d\mathbf{x}' \equiv dx'_1 dx'_2 \cdots dx'_n$,

$$\begin{split} \Psi_1(\mathbf{x} - \mathbf{x}') &\equiv -\frac{1}{2} |x - x'| , \\ \Psi_2(\mathbf{x} - \mathbf{x}') &\equiv -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}'| , \end{split}$$

and for $n \geq 3$

$$\Psi_n(\mathbf{x} - \mathbf{x}') \equiv \frac{\Gamma(1 + n/2)}{n(n-2)\pi^{n/2}} |\mathbf{x} - \mathbf{x}'|^{2-n} .$$

Here

$$\Gamma(1+n/2) = \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}{2^n} \sqrt{\pi}$$

is the Gamma Function

$$\Gamma(z) \equiv \int_0^\infty dt \ t^{z-1} e^{-t}$$

evaluated at half-integers (i.e., n is odd in this expression). When the Gamma Function is evaluated at the integers, it is just the factorial function:

$$\Gamma(1+n) = n\Gamma(n) = n!$$

(A) Verify that Ψ_3 results in the expression for the gravitational potential quoted in the text.

(B) For n = 3 consider an $f(\mathbf{x})$ which is independent of x_3 , but which depends in some fashion upon x_1 and x_2 . Convince yourself that the resulting ψ cannot depend upon x_3 either. Then, derive an expression for $\psi(x_1, x_2)$ by integrating over the ignorable x'_3 coordinate.

Exercise 2: SPHERICALLY-SYMMETRIC MASS DISTRIBUTIONS

(A) Find the gravitational potential corresponding to the spherically-symmetric distribution of matter, of radius a and total mass M, given by

$$\rho(r) = \frac{3(\alpha+3)M}{4\pi\alpha a^{3+\alpha}} (a^{\alpha} - r^{\alpha}) ,$$

for $0 \le r \le a$, $\rho(r) = 0$ for $r \ge a$, where $\alpha > 0$ is a non-negative constant.

(B) Discuss the limiting case $\alpha \to \infty$.

(C) Discuss the limiting case $\alpha \to 0$.

Exercise 3: A VERY USEFUL REPLACEMENT

Although the first exercise gives wonderfully compact and elegant solutions to Poisson's Equation in any number of spatial dimensions, in practice it is much more advantageous to replace the quantity $|\mathbf{x} - \mathbf{x}'|$ by something that involves \mathbf{x} and \mathbf{x}' separately. For n = 3, this *very* useful replacement is

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(-1)^m}{2l+1} \frac{|\mathbf{x}_{<}|^l}{|\mathbf{x}_{>}|^{l+1}} Y_l^{-m}(\theta', \phi') Y_l^m(\theta, \phi) ,$$

where the greater of ${\bf x}$ and ${\bf x}'$ goes in the denominator and the lesser in the numerator. The

$$Y_l^m(\theta,\phi) \equiv \sqrt{\frac{21+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \exp im\phi \ ,$$

are the spherical harmonics, and the $P_l^m(x)$ are the associated Legendre Functions, defined as

$$P_l^m(x) \equiv (1-x^2)^{m/2} \frac{(-1)^m}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l ,$$

valid for $|x| \leq 1$. This result shows that projection of $\rho(\mathbf{x}')$ onto a given spherical harmonic generates a complementary spherical harmonic component to the resulting gravitational potential!

(A) Find the potential outside of a slightly flattened sphere of constant density, and total mass M, whose surface radius is given by $a + bP_2(\cos \theta)$. Show that in the limit $b \to 0$ you recover the solution to exercise 2(B) above.

(B) Find, or derive, the very useful replacement for n = 2.

Exercise 4: <u>DISTRIBUTIONS AND GENERALIZED FUNCTIONS</u> In a purely "operational" sense, Exercise 1, suggests that

$$\nabla^2 \Psi_n(\mathbf{x} - \mathbf{x}') \quad " = " - \delta(\mathbf{x} - \mathbf{x}') \equiv -\delta(x_1 - x_1')\delta(x_2 - x_2')\cdots\delta(x_n - x_n')$$

where the *Dirac delta function*, has the property that,

$$\int dx' f(x')\delta(x-x') = f(x)$$

for almost all sensible functions f. In fact $\delta(x)$ is not really a function, but instead should be thought of as a *distribution*, since its meaning *outside* of an integral is somewhat pathological. As we demonstrated in Section 2, for n = 3, $\delta(\mathbf{x})$ is precisely zero everywhere *except* at $|\mathbf{x}| = 0$ where it is not actually defined.

One (but not the only) way to give the delta function some semblance of meaning outside of an integral is to think of it as the limiting end state of a sequence of functions that are well-behaved everywhere. We can associate with the limit of the sequence a *distribution*.

(A) Consider the sequence of functions

$$S_m(x) = \{\exp(-x^2/n^2)\}_{n=1}^{n=m}$$
,

where n and m are positive integers, and the corresponding sequence of integrals

$$\mathcal{F}_m = \left\{ \int_{-\infty}^{\infty} dx \ F(x) \exp(-x^2/n^2) \right\}_{n=1}^{n=m}$$

Now take the limit $m \to \infty$, the limit of the sequence $S_{m\to\infty}$ is the generalized function, or distribution, which we can call I(x) = 1, and the limit of the sequence of integrals is just

$$\int_{-\infty}^{\infty} dx \ F(x)I(x) = \int_{-\infty}^{\infty} dx \ F(x) \ .$$

Find a different sequence that yields the same limit. (B) Now consider the sequence of functions

$$S_m(x) = \{\sqrt{n/\pi} \exp(-nx^2)\}_{n=1}^{n=m}$$
,

where n and m are positive integers, and the corresponding sequence of integrals

$$\mathcal{F}_m = \left\{ \sqrt{n/\pi} \int_{-\infty}^{\infty} dx \ F(x) \exp(-nx^2) \right\}_{n=1}^{n=m} .$$

The limit $m \to \infty$, the limit of the sequence $S_{m\to\infty}$ is the generalized function, or distribution, which has the same properties as our $\delta(x)$. Now show that the corresponding limit of the sequence of integrals is just

$$\int_{-\infty}^{\infty} dx \ F(x)\delta(x) = \ F(0) \ .$$

[Hint: Look at the limit as $n \to \infty$ of

$$\left|F(0) - \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} dx F(x) \exp(-nx^2)\right| \le \left|\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} dx [F(x) - F(0)] \exp(-nx^2)\right| ,$$

and demonstrate that the upper bound on the right goes to zero like $n^{-1/2}$.] Find a different sequence that yields the same limit. (C) Now use the identity

$$F'(0) = \int dx \ F'(x)\delta(x) = \int dx \frac{d}{dx} \left[F(x)\delta(x)\right] - \int dx F(x)\delta'(x) \ ,$$

to convince yourself that the n-the derivative of the Dirac delta function is another generalized function with the property that

$$(-1)^n F^{(n)}(0) = \int_{-\infty}^{\infty} dx \ F(x) \delta^{(n)}(x) \ .$$

Find a sequence that results in $\delta^{(n)}(x)$ as its limit.

(D) Use the identity in the opposite direction to find a function $\theta(x)$, whose derivative with respect to x is the delta function $\delta(x)$. The integration constant is usually fixed by requirement that

$$\theta(x) = 0$$
, for $x < 0$,

and is called Heaviside's Step Function.

7. Further Reading

There are no monographs or review articles on the continuity equation! And after browsing through this Scene you know why.

For gravitation and the Newton-Poisson Equation I have relied mostly upon the treatment in Kulsrud $[{\bf K}~{\bf 1}]$ and

*[**BT 1**] James Binney & Scott Tremaine, <u>Galactic Dynamics</u>, (Princeton, NJ: Princeton University Press; 1987), xv+733,

[S 7] Donald G. Saari, <u>Collisions, Rings, and Other Newtonian N-Body Problems</u>, (Providence, RI: American Mathematical Society; 2006), x+235,

[HH 1] Douglas Heggie & Piet Hut, The Gravitational Million-Body Problem.

A Multidisciplinary Approach to Star Cluster Dynamics, (Cambridge, UK: Cambridge University Press; 2003), xiv+357.

All three of these books are brilliant, and written in a style I have tried unsuccessfully to emulate here. I debated whether to include some aspect of this material, but, given that they all did such a great job, and that it borders on microphysics—although it is some stretch to think of the evolution of a globular cluster with 10^6 stars as a microsphysical problem—I set it aside.

The articles in

[HI 1] Stephen Hawking & Werner Israel, eds., <u>300 Years of Gravitation</u>, (Cambridge, UK: Cambridge University Press; 1989), xiii+690,

especially the remarkable contribution of Thibault Damour, open several vistas on how one might improve markedly upon the treatment of gravitation provided in this *Opera*.

The Lagrangean approach to magnetohydrodynamics is handled very nicely by three recent papers,

*[KD 1] R. Keppens & T. Demaerel, "Stability of ideal MHD configurations. I. Realizing the generality of the *G* operator", *Physics of Plasmas*, 23, 122117, 2016. https://doi.org/10.1063/1.4971811,

*[DK 1] T. Demaerel & R. Keppens, "Stability of ideal MHD configurations. II. Results for stationary equilibrium configurations", *Physics of Plasmas*, 23, 122118, 2016. https://doi.org/10.1063/1.4971812,

 $\star [{\bf 0} \ {\bf 1}]$ Gordon I. Ogilvie, "Astrophysical fluid dynamics", Journal of Plasma Physics, 82(3), 205820301, 2016. https://doi.org/10.1017/S0022377816000489, while the Lagrangean approach to radiation hydrodynamics is discussed by Mihalas & Mihalas [MM 1].

For even more fun with Pfaffians, see

*[C 1] Brian J. Cantwell, Introduction to Symmetry Analysis, (Cambridge, UK: Cambridge University Press; 2002), xli+612,

 $\star [\mathbf{I}\ \mathbf{2}]$ E.L. Ince, Ordinary Differential Equations, (New York, NY: Dover Publications; 1955), $\overbrace{\mathrm{viii}+558}^{\mathrm{viii}+558}$

as well as the terse but incredibly insightful treatment in $11 \text{ of } \mathbf{P1}$ Wolfgang Pauli, Thermodynamics and Kinetic Theory of Gases, The Pauli Lectures on Physics, Volume 3, (Mineola, NY: Dover Publications; 2000), x+138.

The MURaM code and simulations are described by

★[VSSCTL 1] A. Vögler, S. Shelyag, M. Schüssler, F. Cattaneo, T. Emonet & T. Linde, "Simulations of magneto-convection in the solar photosphere. Equations, methods and results of the MURaM code," *Astronomy & Astrophysics*, **429**, 335-51, 2005. https://doi.org/10.1051/0004-6361:20041507

The very "useful replacement" can be found in many place, I used

*[J 1] J.D. Jackson, <u>Classical Electrodynamics</u>, (New York, NY: John Wiley & Sons; 1963), xvii+641,

which is the first edition. Subsequent editions exist and became thicker but often less transparent as more and more topics were added. Also useful in this regard are two rather valuable additions to you mathematical physics collection: [**B** 1] G. Barton, <u>Elements of Green's Functions and Propagation</u>. Potentials, Diffusion and Waves, (Oxford, UK: Clarendon Press; 1995), xiii+465,

[**R 1**] Paul I. Richards, <u>Manual of Mathematical Physics</u>, (New York, NY: Pergamon Press; 1959), xi+486.

In particular, Richards $[\mathbf{R} \ \mathbf{1}]$ is just overflowing with some of the most interesting, curious and at times bizarre relationships.

A particularly clear, careful and useful introduction to distributions and generalized functions can be had from the little book,

*[L 1] M.J. Lighthill, Fourier Analysis and Generalised Functions, (Cambridge, UK: Cambridge University Press; 1978), viii+79,

and then when you are ready to really impress your friends and colleagues with your fluency in distributional matters, devour

[Z 1] A.H. Zeemanian, Distribution Theory and Transform Analysis. An Introduction to Generalized Functions with Applications, (New York, NY: Dover Publications; xii+371.

On the thorny matter of units, the two articles by Frank Wilczek are wonderful for setting the stage for further discussions,

[W 2] Frank Wilczek, "On absolute units, I: Choices, II: Challenges and Response", *Physics Today*, **58**(10)/**59**(1), 12-3/10-1, 2005.

8. Appendix A: Units, Dimensions and all That

We shall need to quantify four things: *mass* (both inertial and gravitational, which happily turn out to be the same, so far as one can tell), *distance* (or lengths), *time* (or duration), and *temperature*. Everything else turns out to be some combination of these four dimensional quantities. For example, velocity is a distance per unit of time, momentum is a mass times a distance per unit of time, volume is the cube of a distance, angular momentum and action are both momentum times a distance, and so forth. Even electric charge, as we shall presently see, is some combination of these four fundamental dimensions.

The four standard units one chooses to employ are entirely irrelevant for all intents and purposes. That's reassuring but it also leads to confusion since fundamental constants of nature, such as the speed of light c, Newton's Constant of gravitation G, Planck's Constant h, and Boltzmann's Constant k_B take on different sizes in each system of dimensions.

We will adopt the centimeter-gram-second-degree-Kelvin system of units in these notes. This is not very different than the more popular (these days) International System of Units, which employ the meter-kilogram-second-degree-Kelvin quartet. Greater differences show up in the treatment of the electromagnetic equations, were we shall employ the Gaussian-cgs-degree-Kelvin system. More on that later.

One may of course wonder whether the physics has a different idea of what the correct set of units are for measuring these four fundamental dimensions. Toward that end, notice that

$$\begin{split} c &= 2.9979 \times 10^{10} \ \mathrm{cm} \ \mathrm{sec}^{-1} \ , \\ G &= 6.6726 \times 10^{-8} \ \mathrm{gm}^{-1} \ \mathrm{cm}^3 \ \mathrm{sec}^{-2} \ , \\ h &= 6.6261 \times 10^{-27} \ \mathrm{gm} \ \mathrm{cm}^2 \ \mathrm{sec}^{-1} \ , \\ k_B &= 1.3807 \times 10^{-16} \ \mathrm{gm} \ \mathrm{cm}^2 \ \mathrm{sec}^{-2} \ \mathrm{deg} \ \mathrm{K}^{-1} \ . \end{split}$$

These four fundamental constants (expressed here in cgs-degree-K units) can be combined uniquely to determine a fiducial mass,

$$m_P \equiv \left(\frac{\hbar c}{G}\right)^{1/2} = 2.176 \times 10^{-5} \text{ gm} ,$$

length

$$l_P \equiv \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.616 \times 10^{-33} \text{ cm}$$

and time

$$t_P \equiv \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.391 \times 10^{-44} \text{ sec}$$

Here, $\hbar \equiv h/2\pi$. The factor of 2π is in some sense arbitrary, but keeps these quantities, referred to as the Planck Mass, Planck Length, and Planck Time, consistent with common usage. Although the Planck Mass is not too cumbersome for day-to-day activities, the Planck Length and the Planck Time are completely impractical. Since the temperature unit appears only in Boltzmann's constant, the analogous unit of temperature is

$$T_P \equiv \left(\frac{\hbar c^5}{Gk_B^2}\right)^{1/2} = 1.417 \times 10^{32} \text{ deg K}$$

the Planck Temperature, which is also not helpful for weather forecasting purposes.

These four fundamental constants do not refer to any particular type of material. The electron mass could have been used as a fundamental mass scale. But then what do we make of the smaller yet neutrino masses? The unit of electric charge might also have been used in place of one of these four constants. But, as we shall see, in the Gaussian-cgs-degree-Kelvin system, the square of the fundamental electric charge, denoted here by e, is simply proportional to $\hbar c$,

$$e^2 = \frac{1}{137.03599...}\hbar c \text{ gm cm}^3 \text{ sec}^{-2}$$
,

so we are back to the same set of fundamental constants. Indeed, this is one of the attractions of the Gaussian-cgs-degree-Kelvin system of units.

The numerical factor in front of $\hbar c$ is called the *fine structure constant*, α :

$$\alpha \equiv \frac{e^2}{\hbar c} \approx \frac{1}{137} \; ,$$

which should be compared with

$$G\frac{m_e^2}{\hbar c}\approx 1.75\times 10^{-22}~,$$

where m_e is the mass of the electron, which might be regarded as a *really* fine structure constant. This indicates that our two long-range forces, gravity and electromagnetism, are of quite different strengths for the material that populates our *Opera*, with profound implications.

Finally, a choice of units that really cleans up notation selects a mass, length, time and temperature so that the equivalent values of c, G, h and k_B are all 1 in these units. So in these units, $\hbar = 1/2\pi$, which is the ratio of radius to the circumference of a circle. (!) The disadvantage of this approach is that it is much harder to carry out dimensional analysis on an equation as a check of one's work.

9. Appendix B: Spherical Geometry

In spherical coordinates (r, θ, ϕ) the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \rho u_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho u_\phi) = 0 \; .$$

The components of the gravitational field \mathbf{g} are

$$g_r = -\frac{\partial \Phi}{\partial r}, \qquad g_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \qquad g_\phi = -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}$$

and Poisson's Equation for the gravitational potential Φ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 4\pi G\rho$$

It sometimes proves advantageous to replace the polar angle θ by $\mu = \cos \theta$ or $\sin \theta = \sqrt{1 - \mu^2}$, using the chain rule

$$\frac{\partial}{\partial \theta} = \frac{\partial \mu}{\partial \theta} \frac{\partial}{\partial \mu} = -\sqrt{1-\mu^2} \frac{\partial}{\partial \mu} \; .$$

This has a tendency to simplify the appearance of some of the expressions above.

We don't bother with similar expressions in cylindrical coordinates because, the last time we checked, there were no cylindrical planets, stars, clusters or galaxies. There are, however, highly flattened spheres or disks, and in such circumstances, *oblate spheroidal coordinates* can be effective, albeit, much harder to use than spherical coordinates.

10. Appendix C: Einstein Summation Convention

When an italicized Latin subscript is repeated in an expression, we understand that there is an implied summation over all the possible values that index can take on. For example,

$$U_i \frac{\partial \Psi}{\partial x_i} \equiv \sum_i U_i \frac{\partial \Psi}{\partial x_i} \equiv U_1 \frac{\partial \Psi}{\partial x_1} + U_2 \frac{\partial \Psi}{\partial x_2} + \cdots$$

In general relativity and differential geometry one must distinguish between covariant and contravariant vectors (which live in a vector space and its dual vector space, respectively) by employing such indices as subscripts or superscripts (respectively). For the most part we will ignore such fine points in these notes and usually keep indices as subscripts to avoid confusion with powers.

Two particularly useful quantities that arise in these matters are the Kronecker Delta δ_{ij} , which is 1 if i = j and 0 otherwise; and the Levi-Civita Symbol ϵ_{ijk} , which is 1 if ijk is an even permutation of 123, -1 if ijk is an odd permutation of 123, and 0 if any two indices are equal. The following results often prove useful in simplifying expressions

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} ,$$

$$\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il},$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6 .$$

11. Appendix D: Eulerian vs Lagrangean Approaches to RMHD

The developments of §3 are indicative of the Lagrangean viewpoint of fluid dynamics. The essential concept is that of a unique continuous (or as continuous as possible) family of mappings \mathbf{X} from the *intial* Eulerian location of a fluid element \mathbf{x}' at time t = 0 to a *future* Eulerian position $\mathbf{x} = \mathbf{X}(\mathbf{x}', t)$ at any later time $t \ge 0$. Encoded in this family of mappings is all the dynamical information needed about the astrophysical system.

For example, in §3, we demonstrated how knowledge of the family of mappings, via the Jacobian, provides the Lagrangean density at a future time in terms of the initial density, consistent with the conservation of mass,

$$\rho(\mathbf{x}',t) = \frac{\rho_0(\mathbf{x}')}{J(\mathbf{x}',t)} = \frac{\rho(\mathbf{x}',0)}{J(\mathbf{x}',t)}$$

The magnetic induction equation in MHD,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \; ,$$

which expresses the conservation of magnetic flux through a fluid element, similarly has a very simple and elegant solution—the so-called *Cauchy* solution from the Lagrangean viewpoint

$$B_k(\mathbf{x}',t) = \frac{B_j(\mathbf{x}',0)}{J(\mathbf{x}',t)} \frac{\partial X_k}{\partial x_i'},$$

where $\mathbf{B}_0(\mathbf{x}') = \mathbf{B}(\mathbf{x}', t = 0)$ is the initial magnetic field.

The equation for the family of mappings, is however, rather complicated and is no easier to solve than the full set of RMHD equations in the Eulerian framework. One advantage, as we shall later see, is that the Lagrangean framework permits us to formulate the solution in terms of a *variational principle*, which has certain practical and conceptual advantages. [See Act II Scene 2. Appendix A.]

12. Appendix E: Pfun with Pfaffians

In this appendix we develop the theory of solving first-order partial differential equations (PDEs). Or, we have some fun with Pfaffians.

$\alpha.$ Two Variables

The simplest formulation involves two variables, x_1 and x_2 , and a first order equation of the form

$$U_1(\mathbf{x})\frac{dx_2}{dx_1} = U_2(\mathbf{x}) \; ,$$

which is equivalent to the single equation

$$\frac{dx_2}{dx_1} = \frac{U_2}{U_1} \equiv u_1(\mathbf{x}) \; .$$

Following Lagrange, we seek a function $\Psi(\mathbf{x})$ which provides an integral of the motion. To select a definite trajectory for the solution we require an additional constraint, namely that x_2 takes a definite value when $x_1 = 0$, say. Let $\mathbf{x}' = [0, x_2(0)]$ describe this constraint. Then

$$\Psi(\mathbf{x}) = \psi_0 = \Psi(\mathbf{x}')$$

provides an implicit solution to the equation.

The equation for Ψ comes from taking the total derivative with respect to x_1 :

$$\frac{d\Psi}{dx_1} = \frac{\partial\Psi}{\partial x_1} + \frac{\partial\Psi}{\partial x_2}\frac{dx_2}{dx_1} = 0$$

which must vanish along the solution trajectory. Replacing dx_2/dx_1 by U_2/U_1 gives

$$U_i \frac{\partial \Psi}{\partial x_i} = 0$$

summed over i = 1, 2. Clearly all three of our equations are equivalent and so saying the characteristic of this equation is given by

$$\frac{dx_1}{U_1} = \frac{dx_2}{U_2}$$

in fact adds nothing new. It simply returns our original equation

$$U_1(\mathbf{x})\frac{dx_2}{dx_1} = U_2(\mathbf{x})$$

There is one final equivalent form which is the Pfaffian form

$$\delta Q = \xi_1(\mathbf{x}) dx_1 + \xi_2(\mathbf{x}) dx_2 = U_2 dx_1 - U_1 dx_2 = 0 \; .$$

This should be compared with the exact differential:

$$d\Psi = \frac{\partial \Psi}{\partial x_1} dx_1 + \frac{\partial \Psi}{\partial x_2} dx_2 = 0 \ .$$

Now it is tempting to try to identify U_2 with $\partial \Psi / \partial x_2$ and integrate this result, but this will only work if

$$\frac{\partial U_2}{\partial x_1} + \frac{\partial U_1}{\partial x_2} = 0 \; .$$

If this is true, the $\delta Q = dQ$ is an exact differential. Otherwise it is not.

Suppose its not. Then Pfaff's Theorem assures us that there exists a function $\tau(\mathbf{x})$ which we can divide δQ by to obtain an exact differential:

$$\frac{\delta Q}{\tau} = d\Psi = \frac{U_2}{\tau} dx_1 - \frac{U_1}{\tau} dx_2 = 0$$

We call this an integrating factor. That's the good news. The bad news is that if we try to solve for τ we end up back where we started needing to solve any one of the above equivalent equations. So there is no general solution we can write down that holds for arbitrary $u_2(\mathbf{x})$, period.

This leaves us with two options. We can try to guess an integrating factor, or see if someone else has managed to guess one for us. We can try to look for symmetry transformations of \mathbf{x} which leave the equations invariant. The latter approach involves Lie Groups and symmetry analysis which is perhaps the only systematic means to solve nonlinear equations. The former, often the result of successfully applying the latter, requires access to a tabulation of previous results.

We close with a few interesting results from the world of look up tables. The cases

$$u_1(\mathbf{x}) = f(x_1)x_2^{\alpha} + g(x_1)$$
 and $u_1(\mathbf{x}) = [f(x_2)x_1^{\alpha} + g(x_2)]^{-1}$

result in a linear first order ODE's, for arbitrary functions f and g and constant α . For $\alpha \neq 1$ this equation is known as Bernoulli's equation and is reducible to a linear ODE with the substitution $y = x^{1-\alpha}$. Another famous example is Riccati's Equation

$$u_1(\mathbf{x}) = f(x_1)x_2^2 + g(x_1)x_2 + h(x_1)$$
 and $u_1(\mathbf{x}) = [f(x_2)x_1^2 + g(x_2)x_1 + h(x_2)]^{-1}$

for arbitrary functions f, g and h. The Riccati transformation removes the nonlinearity but increases the order of the equation by 1. Another solvable case is $u_1(\mathbf{x}) = f(x_1/x_2)$, which obviously has a high degree of symmetry.

Notice that $U_i(\mathbf{x})$ could be wildly complicated so long as their ratio $u_1(\mathbf{x})$ is amenable to progress! So having all these equivalent forms is powerful.

β . Three Variables

We add x_3 into the mix. Now

$$U_1(\mathbf{x})\frac{\partial x_3}{\partial x_1} + U_2(\mathbf{x})\frac{\partial x_3}{\partial x_2} = U_3(\mathbf{x}) ,$$

is our basic partial differential equation. Lagrange now asserts that we may expect *two* integrals of the motion, say $\Psi^1(\mathbf{x})$ and $\Psi^2(\mathbf{x})$ which will both be solutions of the same equation. This equation is derived as before. For either Ψ

$$\frac{d\Psi}{dx_1} = \frac{\partial\Psi}{\partial x_1} + \frac{\partial\Psi}{\partial x_3}\frac{\partial x_3}{\partial x_1} = 0 ,$$

and

$$\frac{d\Psi}{dx_2} = \frac{\partial\Psi}{\partial x_2} + \frac{\partial\Psi}{\partial x_3}\frac{\partial x_3}{\partial x_2} = 0 ,$$

which together imply

$$U_i \frac{\partial \Psi}{\partial x_i} = 0 \; ,$$

with our sum now extending over i = 1, 2, 3. The characteristic equations are

$$\frac{dx_1}{U_1} = \frac{dx_2}{U_2} = \frac{dx_3}{U_3} \; .$$

Which are equivalent to a system of evolution equations

$$\frac{dx_2}{dx_1} = \frac{U_2}{U_1} \equiv u_1(\mathbf{x}) ,$$
$$\frac{dx_3}{dx_1} = \frac{U_3}{U_1} \equiv u_2(\mathbf{x}) ,$$
$$\frac{dx_3}{dx_2} = \frac{U_3}{U_2} = \frac{u_2(\mathbf{x})}{u_1(\mathbf{x})} .$$

of which only two are independent.

It is worth pointing out that if the u_i or their ratio can be made independent of one of the three variables then the system is autonomous and can be reduced to solving the two variable problem above. Otherwise we can make a best effort to try to solve this system of equations, which will provide two integrals of the motion $\Psi^1(\mathbf{x})$ and $\Psi^2(\mathbf{x})$. The general solution is an arbitrary function of these two integrals, $\Psi(\mathbf{x}) = \omega(\Psi^1, \Psi^2)$.

The Pfaffian approach rewites this system of equations as

$$U_1 dx_2 - U_2 dx_1 = 0 ,$$

$$U_1 dx_3 - U_3 dx_1 = 0 ,$$

$$U_2 dx_3 - U_3 dx_2 = 0$$

and takes some linear combination to form

$$\delta Q = \xi_i(\mathbf{x}) dx_i \; .$$

An integrating factor $\tau(\mathbf{x})$ is not guaranteed unless

$$\epsilon_{ijk}\xi_i \frac{\partial \xi_k}{\partial x_j} = \boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\xi} = 0 \; .$$

Therefore, we must form our superposition of the these three equations in such a fashion that this criterion is fulfilled. There are two independent ways to accomplish this, and each, if an integrating factor can be identified, generates one of the two Ψ functions.

 γ . Three Variables: An Illustrative Example

To illustrate how this all works, *when it works*, we consider the following simple example

$$(x_2+x_3)\frac{\partial x_3}{\partial x_1}+(x_1+x_3)\frac{\partial x_3}{\partial x_2}=x_1+x_2,$$

or

$$(x_3 + x_2)\frac{\partial\Psi}{\partial x_1} + (x_3 + x_1)\frac{\partial\Psi}{\partial x_2} + (x_1 + x_2)\frac{\partial\Psi}{\partial x_3} = 0.$$

This system has a lot of symmetry, so it is not surprising that it is going to yield an analytic result. The characteristic equations are

$$\frac{dx_1}{x_2 + x_3} = \frac{dx_2}{x_1 + x_3} = \frac{dx_3}{x_2 + x_1} ,$$
$$dx_1 \qquad x_2 + x_3$$

or

$$\frac{dx_1}{dx_3} = \frac{x_2 + x_3}{x_2 + x_1} ,$$
$$\frac{dx_2}{dx_3} = \frac{x_1 + x_3}{x_2 + x_1} .$$

Our next step is to try to find a way to integrate them. Some noodling around suggests subtracting these two equations, to get:

$$\frac{d}{dx_3}(x_1 - x_2) = \frac{x_2 - x_1}{x_2 + x_1} \; .$$

This is encouraging because we can divide through and get the combination $x_2 - x_1$ alone on one side of this equation, leaving the factor $(x_2 + x_1)^{-1}$ alone on the other side. If we can find another way to come up with something similar in structure we can make some progress.

So we next we subtract 1 from the first equation

$$\frac{dx_1}{dx_3} - 1 = \frac{x_3 - x_1}{x_2 + x_1} = \frac{d}{dx_3}(x_1 - x_3) ,$$

and make use of the obvious, but quite sneaky result that $1 = dx_3/dx_3$. Now we are set because

$$\frac{1}{x_1 + x_2} = \frac{1}{x_3 - x_1} \frac{d}{dx_3} (x_1 - x_3) = \frac{1}{x_2 - x_1} \frac{d}{dx_3} (x_1 - x_2)$$

implies

$$\frac{d}{dx_3}\log\frac{x_1 - x_3}{x_1 - x_2} = 0$$

so our first integral of the motion is

$$\Psi^1(\mathbf{x}) = \frac{x_1 - x_3}{x_1 - x_2} \; .$$

Can we get to the same outcome by still another way? Yes. By adding 1 to $dx_1/dx_3 + dx_2/dx_3$ we find

$$\frac{d}{dx_3}(x_1 + x_2 + x_3) = 2\frac{x_1 + x_2 + x_3}{x_1 + x_2}$$

 \mathbf{SO}

$$\frac{1}{x_1 + x_2} = \frac{1}{2} \frac{1}{x_1 + x_2 + x_3} \frac{d}{dx_3} (x_1 + x_2 + x_3) = \frac{1}{x_3 - x_1} \frac{d}{dx_3} (x_1 - x_3) = \frac{1}{x_2 - x_1} \frac{d}{dx_3} (x_1 - x_2)$$

which provides our second integral of the motion

$$\frac{d}{dx_3}\log\frac{x_1+x_2+x_3}{(x_1-x_3)^2} = 0$$

or

$$\Psi^2(\mathbf{x}) = \frac{x_1 + x_2 + x_3}{(x_1 - x_3)^2} \; .$$

We could equally well put $(x_1 - x_2)^2$ in the denominator instead and we still have a second independent constant of the motion because we can multiply this expression by the square of Ψ^1 — or any other function of Ψ^1 for that matter.

Now as the constant ψ_1 varies, the equation

$$\Psi^1(\mathbf{x}) = \frac{x_1 - x_3}{x_1 - x_2} = \psi_1$$

traces out a series of two-dimensional laminated surfaces. If we know that our trajectory passes through the point $\mathbf{x}' = [x'_1, x'_2, x'_3]$ we can evaluate the constant $\psi_1 = \Psi^1(\mathbf{x}')$. A similar result holds for Ψ^2 and the intersection of these two surfaces is a one-dimensional curve satisfying the pair of equations

$$\frac{x_1 - x_3}{x_1 - x_2} = \psi_1 ,$$
$$\frac{x_1 + x_2 + x_3}{(x_1 - x_3)^2} = \psi_2 .$$

The first equation describes a plane that passes through the origin

$$(1-\psi_1)x_1+\psi_1x_2-x_3=0.$$

The second allows us to solve explicitly for

$$x_2 = -(x_1 + x_3) + \psi_2(x_1 - x_3)^2$$

Using this in the equation for the plane, we obtain a single equation for the remaining two variables:

$$\psi_1\psi_2(x_1-x_3)^2 + (1-\psi_1-\psi_2)x_1 - (1+\psi_2)x_3 = 0$$

which is the equation for a parabola that passes through the origin.

The parabolic nature of the solution is clear in a coordinate system rotated by $\pm \pi/4$ so that $\xi = x_1 - x_3$ and $\eta = x_1 + x_3$ serve as the orthogonal coordinates. In terms of ξ and η this last equation reads:

$$\xi = \frac{1}{\psi_1 + 2\psi_2} \eta(\psi_1 \psi_2 \eta + 1 - \psi_2) \; .$$

Using

$$x_{1} = \frac{1}{2}(\xi + \eta) = \frac{1}{2(\psi_{1} + 2\psi_{2})}\eta(\psi_{1}\psi_{2}\eta + 1 + \psi_{1} + \psi_{2}) ,$$

$$x_{3} = \frac{1}{2}(\eta - \xi) = -\frac{1}{2(\psi_{1} + 2\psi_{2})}\eta(\psi_{1}\psi_{2}\eta + 1 - \psi_{1} - 3\psi_{2}) ,$$

$$x_{2} = -\frac{1}{\psi_{1} + 2\psi_{2}}\eta(2\psi_{2}^{2}\eta + 1 - \psi_{2}) ,$$

we recover the complete solution in parametric form where $\eta = x_1 + x_3$ varies from $-\infty$ to $+\infty$. All three x_i are quadratic in η with x_1 and x_3 headed in opposite directions for large η . The ratio x_i/x_j tends to a constant for large η , and $|x_i|$. Every solution, no matter where \mathbf{x}' is located must pass through the origin $\mathbf{x} = 0$. The solution vector \mathbf{x} lies in a two-dimensional plane that contains the origin and is oriented by the value of ψ_1 . That is, the solution is contained entirely in a two-dimensional subspace, or manifold, of the full three dimensional Euclidean vector space.

We can now go back and verify directly by substitution that this parametric solution indeed solves

$$(x_2 + x_3)\frac{\partial x_3}{\partial x_1} + (x_1 + x_3)\frac{\partial x_3}{\partial x_2} = x_1 + x_2 ,$$

and that when $\eta = x'_1 + x'_3$, and using $\psi_1 = (x'_1 - x'_3)/(x'_1 - x'_2)$ and $\psi_2 = (x'_1 + x'_2 + x'_3)/(x'_1 - x'_2)^2$ this system ensures that $x_i = x'_i$ for i = 1, 2, 3. We can also verify directly that $\Psi^1(\mathbf{x})$ and $\Psi^2(\mathbf{x})$ are each solutions of

$$(x_3 + x_2)\frac{\partial\Psi}{\partial x_1} + (x_3 + x_1)\frac{\partial\Psi}{\partial x_2} + (x_1 + x_2)\frac{\partial\Psi}{\partial x_3} = 0.$$

A lot of algebra and a bit of luck to be sure but hopefully this gives a concrete sense of how this all fits together.

δ . Even More Variables

It should now reasonably clear how we can generalise this to four or more variables (recall that our continuity equation is the case of five variables, $x_4 = t$, $x_5 = \rho$, which is a tough way to start out).

Going to four, for example, simply adds a new variable x_4 , an additional term involving U_4 in the canonical partial differential equation, an additional characteristic equation, an additional u_3 , an additional Ψ^3 and so forth. In a very very nice twist of fate the guarantee of there being an integrating factor for the Pfaffian (now with an additional $\xi_4 dx_4$ term) remains unchanged

$$\epsilon_{ijk}\xi_i\frac{\partial\xi_k}{\partial x_j} = 0$$

for all admissible combinations of ijk. When ijk could only take on the values 1, 2, and 3 there was only one nontrivial admissible combination, 123, and therefore one constraint. With 4 choices there are three conditions corresponding to 123, 124, and 234. In general, for n variables there are (n-1)(n-2)/2 constraints needed to guarantee the existence of an integrating factor.

12. Appendix F: RMHD's 58 Terms

The mass density is

 ρ

and the mass flux is

 $\rho \mathbf{u}$.

There are no exchange terms. Two down, 56 to go.