Appendix C: RMHD Equilibria

T.J. Bogdan

September 22, 2018

It had of course been rumored for some time that there was a mssing Scene from Act II that helped to bridge the perplexing (to critics at least) gap in the plot. This manuscript was recently found by archivists rummaging around in a dusty library basement. It's authenticity, however, remains uncertain.
—The Publishers

1. Introduction

In this Scene we set about constructing static, steady-state RMHD equilibria. All time-derivatives $\partial/\partial t \equiv 0$ are set to zero, as well as the fluid velocity $\mathbf{u} \equiv 0$. This implies that the continuity equation is out the window because all it contains are time derivatives and velocities. All of Navier's and Stokes's hard work on viscous dissipation is not needed, nor is Burger's steepening studies. Oh, and all this effort to work in the comoving frame is accomplished with no effort, as, the comoving frame *is* the laboratory frame!

What. if anything, is left?

2. The Equations of Radiation Magnetohydrostatics

The equations for gravity,

$$\nabla^2 \Phi = 4\pi G \rho \ ,$$

$$\nabla \cdot \mathbb{G} = \rho \nabla \Phi \ ,$$

electromagnetism,

$$\mathbf{J} = \sigma \mathbf{E} \ ,$$

$$c \nabla \times \mathbf{B} = 4\pi \mathbf{J} \ ,$$

$$c \nabla \times \mathbf{E} = 0 \ ,$$

$$\nabla \cdot \mathbf{E} = 4\pi \delta \ ,$$

$$\nabla \cdot \mathbf{B} = 0 \ ,$$

$$\nabla \cdot (\sigma \mathbf{E}) = 0 \ ,$$

$$\nabla \cdot \mathbb{M} = -\delta \mathbf{E} - \frac{1}{c} \mathbf{J} \times \mathbf{B} \ ,$$

$$\nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \ ,$$

radiation,

$$\eta_\nu(\mathbf{n}) = \kappa_0(\nu) B_\nu[T] + \sigma_0(\nu) J_\nu \ ,$$

$$\chi_\nu(\mathbf{n}) = \kappa_0(\nu) + \sigma_0(\nu) \ ,$$

$$\mathbf{n}\cdot\nabla I_\nu = \eta_\nu - \chi_\nu I_\nu \ ,$$
$$\nabla\cdot\mathbf{F} = \int_0^\infty d\nu \oint d\mathbf{n}\ [\eta_\nu - \chi_\nu I_\nu] \ ,$$
$$\nabla\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu \oint d\mathbf{n}\ \mathbf{n}[\eta_\nu - \chi_\nu I_\nu] \ ,$$

and the material,

$$0 = \rho\frac{\delta q}{\delta t} = \nabla\cdot\Bbbk\cdot\nabla T + \mathbf{J}\cdot\mathbf{E} - \int_0^\infty d\nu \oint d\mathbf{n}\ [\eta_\nu - \chi_\nu I_\nu] \ ,$$
$$\nabla p = -\rho\nabla\Phi + \delta\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B} - \frac{1}{c}\int_0^\infty d\nu \oint d\mathbf{n}\ \mathbf{n}[\eta_\nu - \chi_\nu I_\nu] \ ,$$
$$p = p(\rho, T) \ ,$$

simplify dramatically in absence of material motion and temporal evolution.

From this *extensive* array of identities, *two* essential equations emerge, the *energy equation*:

$$\nabla\cdot(\mathbf{F} - \Bbbk\cdot\nabla T + \mathbf{S}) = 0 \ ,$$

and the *force-balance equation*:

$$\nabla\cdot(p\mathbb{1} + \mathbb{P} + \mathbb{M} + \mathbb{G}) = 0 \ .$$

Let's go to work! For static electromagnetic fields, recall that

$$\mathbf{E} = -\nabla\phi \ , \qquad \mathbf{B} = \nabla\times\mathbf{A} \ .$$

So the charge density and the current density follow immediately from

$$\delta = -\frac{1}{4\pi}\nabla^2\phi \ , \qquad \mathbf{J} = \frac{c}{4\pi}\nabla\times(\nabla\times\mathbf{A}) \ .$$

This leaves one non-trivial relationship (Ampère's Equation) between $\mathbf{E}$ and $\mathbf{B}$, or equivalently ϕ and $\mathbf{A}$:

$$c\nabla\times\mathbf{B} = 4\pi\sigma\mathbf{E} \ , \quad \Longrightarrow \quad c\nabla\times(\nabla\times\mathbf{A}) = -4\pi\sigma\nabla\phi \ .$$

The Lorentz Force is

$$\frac{1}{8\pi}\nabla|\mathbf{E}|^2 + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} = -\nabla\cdot\mathbb{M} \ ,$$

and the Poynting Flux is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \ .$$

Ideal MHD corresponds to the limit $\sigma\to\infty$ which in turn forces $\mathbf{E}\to 0$ while leaving $\mathbf{J}$, $\nabla\times\mathbf{B}$ unconstrained. Ideal EHD corresponds to the limit $\sigma\to 0$, which forces $\mathbf{J}$, $\nabla\times\mathbf{B}$ to zero, which implies

$$\mathbf{B} = -\nabla\psi \ , \ \nabla^2\psi = 0 \ ,$$

and leaves ϕ unconstrained. In between these two extreme cases, we must ensure Ampère's Equation is valid. And the Lorentz Force and the Poynting Flux must be accommodated in the energy and force-balance equations above. We need a closure relationship to determine σ.

For the radiation field,

$$\mathbf{n} \cdot \nabla I_\nu = \kappa_0(\nu) B_\nu[T] + \sigma_0(\nu) J_\nu - [\kappa_0(\nu) + \sigma_0(\nu)] I_\nu \ ,$$

$$\mathbf{n} \cdot \nabla I_\nu = [\kappa_0(\nu) + \sigma_0(\nu)] \left(\frac{\kappa_0(\nu) B_\nu[T] + \sigma_0(\nu) J_\nu}{\kappa_0(\nu) + \sigma_0(\nu)} - I_\nu \right) \ ,$$

defines the source function

$$S_\nu \equiv \frac{\kappa_0(\nu) B_\nu[T] + \sigma_0(\nu) J_\nu}{\kappa_0(\nu) + \sigma_0(\nu)} \ ,$$

which accounts for isotropic thermal emission and scattering. Here we come to an inglorious screeching halt as the right sides of the two moment equations are just nasty integrals over frequency. Svein Rosseland proposed an ingenious way to proceed, by writing these equations as:

$$\nabla \cdot \mathbf{F} = \langle \kappa \rangle [4\sigma_R T^4 - cE] \ ,$$

$$\nabla \cdot \mathbb{P} = -\frac{\langle \kappa \rangle + \langle \sigma \rangle}{c} \mathbf{F} \ ,$$

which would be *exact* if κ_0 and σ_0 were frequency-independent. Of course, they are not, so in practice this cludge involves approximating them by some sort of weighted frequency average and hoping for the best. Realistically, one can also iterate to improve the approximation. Sometimes $\langle \chi \rangle$ is referred to as the *Rosseland-mean opacity.* The various opacity fits provided back in Act II Scene 2 apply to these frequency-averaged quantities, since, as you recall, there was no frequency-dependences in these fits, just temperatures and densities. So someone else has done a lot of the hard work here.

To the same level of approximation, we now solve for the frequency-integrated intensity from

$$\mathbf{n} \cdot \nabla I = [\langle \kappa \rangle + \langle \sigma \rangle] \left(\frac{\langle \kappa \rangle B[T] + \langle \sigma \rangle J}{\langle \kappa \rangle + \langle \sigma \rangle} - I \right) \ ,$$

(which amounts, notationally to simply dropping the ν's) where the moments are

$$J = \frac{c}{4\pi} E = \frac{1}{4\pi} \oint d\mathbf{n} \ I \ ,$$

$$\mathbf{H} = 4\pi \mathbf{F} = \frac{1}{4\pi} \oint d\mathbf{n} \ \mathbf{n} I \ ,$$

and

$$B[T] = \frac{\sigma_R}{\pi} T^4 \ .$$

For the material, there is not much left to do except to decide upon an equation of state, and fix the two transport coefficients, σ and $\Bbbk$. Obviously if we go the partial ionization route we must prepare for a sustained numerical attack on even static and stationary equilibria. So we'll take the ideal gas option,

$$p = (c_p - c_V)\rho T \ ,$$

where c_p and c_V are individually constants.

Finally, there is nothing more to say about gravity than the two equations at the very beginning of this section.

3. Planar Geometry. Part 1

Progress of any sort with the equations of radiative transfer, as you will recall, requires us to work in planar, cylindrical (which of course, I abhor!) or spherical geometries, where we can exploit certain symmetries to reduce the scope of the problem. Planar problems arise when spatial variations in one direction, usually associated with the vertical, are much more pronounced relative to the remaining two (horizontal) directions.

Here it is almost always the case that self-gravitation should be ignored completely. When gravity is present, it should be supplied by external agents and must be a solution of Laplace's Equation

$$\nabla^2 \Phi = 0$$

throughout our astrophysical system. Typically

$$\Phi = gz \ ,$$

with g a strict constant, uniquely satisfies all the necessary requirements, where, $z \equiv x_3$ is the vertical Cartesian coordinate. Neglecting electomagnetism and radiation for the moment, the force-balance equation implies

$$\nabla p = -\rho \nabla \Phi \ , \implies \quad \rho = -\frac{dp}{d\Phi} \ ,$$

The equation of state requires,

$$T = -\frac{1}{c_p - c_V} p \frac{d\Phi}{dp} \ .$$

And finally, the energy equation tells us that

$$\Bbbk \cdot \nabla T = \text{ constant } .$$

From our closure work in Act II Scene 2, we have $\Bbbk \propto T^{5/2}$, for a stellar atmosphere, so

$$\nabla T^{7/2} = \text{ constant } \implies T = (F_0 \Phi + T_0^{7/2})^{2/7}$$

for some constant F_0 which determines the constant energy flux through the atmosphere and another constant T_0 which sets the temperature at the base of our atmosphere.

If we use these results in the equation of state, we get

$$\frac{dp}{p} = -\frac{1}{(c_p - c_V)} \frac{d\Phi}{(F_0\Phi + T_0^{7/2})^{2/7}} \ .$$

This integrates to give

$$p = p_1 \exp\left(-\frac{7}{5}\frac{(F_0\Phi + T_0^{7/2})^{5/7}}{(c_p - c_V)F_0}\right) \ ,$$

in terms of an additional integration constant, p_1. Let, p_0 be the pressure at the base of our planar atmosphere ($z = 0$), then

$$p = p_0 \exp\left(\frac{7}{5}\frac{T_0^{5/2} - (F_0\Phi + T_0^{7/2})^{5/7}}{(c_p - c_V)F_0}\right) \ ,$$

Where T_0 and p_0 are the temperature and pressure at the base of our atmosphere and the constant F_0 accounts for the constant energy flux carried upward through the atmosphere by thermal conduction. This completes our specification of a planar hydrostatic atmosphere! Our atmosphere is essentially isothermal (constant temperature) up to an altitude where

$$\Phi = gz \approx \frac{T_0^{7/2}}{F_0} \ ,$$

and above that point the temperature begins to decline like $z^{2/7}$.

Notice that at great altitudes, $F_0\Phi \gg T_0^{7/2}$, we have

$$p \approx \exp\left(-\frac{7}{5}\frac{g^{5/7}}{(c_p - c_V)F_0^{2/7}} z^{5/7}\right)$$

while near the base of our atmosphere where, $F_0\Phi \ll T_0^{7/2}$, we have

$$p \approx \exp\left(-\frac{g}{(c_p - c_V)T_0} z\right) \ .$$

If we let $F_0 \to 0$, then this last expression obtains everywhere in our fully isothermal atmosphere. So if only thermal conduction was in play, a stellar atmosphere would be isothermal up to some point and then the pressure would fall off more gradually as the conduction sets in.

Suppose, instead we are dealing with the Earth's atmosphere, from the ground up to some location where it is no longer neutral. The analogous expression for κ is considerably more challenging

$$\kappa = \alpha\frac{p}{T} + \beta T + \gamma$$

for some positive constants, α, β, γ. And the equation

$$\Bbbk \cdot \nabla T = \text{ constant },$$

does not just integrate to simply give something. Instead, we get

$$(\alpha \frac{p}{T} + \beta T + \gamma) g \frac{dT}{d\Phi} = -F_0 ,$$

where F_0 is again a positive constant that sets the upward conductive energy flux. The ground is warmed during the day by the sunlight, so the temperature gradient is negative. We also have

$$\frac{dp}{d\Phi} = -\frac{p}{(c_p - c_V)T} .$$

Dividing the two equations to eliminate Φ gives

$$\frac{dp}{dT} = \frac{g}{(c_p - c_V)F_0} \frac{p}{T} \left(\alpha \frac{p}{T} + \beta T + \gamma \right) ,$$

a Pfaffian! Yeah! Which, unfortunately, is just begging for us to find an integrating factor.

What we would like to have the right side of this equation be is a product of the form $f(p)g(T)$. We can force this to happen if we put

$$p(T) = P(T) \cdot T(\gamma + \beta T) ,$$

for a new function $P(T)$. Now

$$\frac{d}{dT} \left[T(\gamma + \beta T)P \right] = \frac{g}{(c_p - c_V)F_0} (\gamma + \beta T)^2 P(\alpha P + 1) ,$$

which does the trick but at the expense of messing up the left side:

$$T(\gamma + \beta T) \frac{dP}{dT} + (\gamma + 2\beta T)P = \frac{g}{(c_p - c_V)F_0} (\gamma + \beta T) P(\alpha P + 1) .$$

This, however, we can fix! Our equation is basically of the form

$$A(T) \frac{dP}{dT} + B(T)P = C(T)P^2 .$$

If we can eliminate one of the three terms then we have an exact differential.

This is accomplished by removing the linear term in P, according to

$$A(T) \exp\left(-\int_{T_0}^{T} dt\ \frac{B(t)}{A(t)} \right) \frac{d}{dT} \left[P \exp\left(\int_{T_0}^{T} dt\ \frac{B(t)}{A(t)} \right) \right] = C(T)P^2 ,$$

and setting

$$Q(T) \equiv P(T) \exp\left(\int_{T_0}^{T} dt\ \frac{B(t)}{A(t)} \right) = \frac{p(T)}{T(\gamma + \beta T)} \exp\left(\int_{T_0}^{T} dt\ \frac{B(t)}{A(t)} \right) ,$$

so that

$$\frac{1}{Q^2}\frac{dQ}{dT} = \frac{C(T)}{A(T)}\exp\left(-\int_{T_0}^{T} dt\ \frac{B(t)}{A(t)}\right)$$

which *is* an exact differential. Here, T_0 is the temperature at the base of the atmosphere ($z = 0$).

The first set of integrations are elementary,

$$\exp\left(\int_{T_0}^{T} dt\ \frac{B(t)}{A(t)}\right) = \frac{\gamma+\beta T}{\gamma+\beta T_0}\left(\frac{T}{T_0}\right)^{1-\frac{\gamma g}{(c_p-c_V)F_0}}\exp\left(\frac{\beta g(T_0-T)}{(c_p-c_V)F_0}\right)$$

and can be done exactly. The remaining integral for $Q(T)$ can also be done exactly in terms of our good friend the Incomplete Gamma Function:

$$\frac{1}{Q^2}\frac{dQ}{dT} = \frac{\alpha g(\gamma+\beta T_0)}{(c_p-c_V)F_0T_0}\left(\frac{T_0}{T}\right)^{2-\frac{\gamma g}{(c_p-c_V)F_0}}\exp\left(\frac{\beta g(T-T_0)}{(c_p-c_V)F_0}\right)\ ,$$

where

$$Q(T_0) = \frac{p_0}{T_0(\gamma+\beta T_0)}\ ,$$

in terms of the atmospheric base pressure, p_0. Note the existence of a critical energy flux given by

$$F_0 = F_{\rm critical} \equiv \frac{\gamma g}{c_p - c_V}\ ,$$

for which the solution behaves quite differently depending upon the energy flux is super- or sub-critical.

Now let's consider the opposite case in which the energy flux through the atmosphere is carried entirely by the radiation field instead of the thermal conduction. This problem we have already solved in Act I Scene 4. We need a constant radiative flux $\mathbf{F} = F_0\hat{\mathbf{e}}_z$ through our atmosphere (so that $\nabla\cdot\mathbf{F} = 0$). The source function is the Planck Function which is also the mean intensity. Therefore we require a solution to Milne's Integral Equation

$$S(\tau) = B[T(\tau)] = J(\tau) = \Lambda_\tau[J(t)]\ ,$$

which carries the requisite energy flux, F_0. This is

$$S(\tau) = B[T(\tau)] = \frac{\sigma_R}{\pi}T^4(\tau) = J(\tau) = \frac{3}{4\pi}F_0[\tau + q(\tau)]\ ,$$

where $q(\tau)$ is the Hopf Function. Strictly speaking, this solution is exact if the optical depth measured into the atmosphere from $z = +\infty$ according to

$$\tau(z) = \int_z^\infty ds\ \langle\chi\rangle(s)$$

satisfies $\tau(0) \equiv \tau_0 \to \infty$. We also have

$$K(\tau) = \frac{1}{4\pi}F_0[\tau + q(\infty)]\ , \qquad H(\tau) = \frac{1}{4\pi}F_0\ .$$

The function $q(\tau)$ has to be one of the strangest objects ever encountered: it starts out at $1/\sqrt{3} = 0.5773...$ at $\tau = 0$ and increases monotonically to $0.710446...$ as $\tau \to \infty$. Go figure. The temperature asymptotes to a minimum value

$$T_\infty = \left(\frac{\sqrt{3} F_0}{4\sigma_R} \right)^{1/4}$$

as $\tau \to 0$ or $z \to \infty$. It increases monotonically with increasing τ reaching a value of

$$T_0 = \left(\frac{3F_0}{4\sigma_R} [\tau + q(\tau)] \right)^{1/4}$$

at the base of our atmosphere at $z = 0$.

This gives us $T(\tau)$. Which is a start. The force balance equation reads

$$\frac{dp}{dz} = -\rho g + \frac{\langle \chi \rangle}{c} F_0 \ .$$

Notice that both ρ and $\langle \chi \rangle$ are functions of z or equivalently, τ. Obviously if F_0 is too big we can end up in a situation where the pressure wants to increase with altitude, and a hydrostatic atmosphere is not achieveable. In any event, the best approach is to work on the optical depth scale, where we at least know $T(\tau)$:

$$\frac{dp}{d\tau} = -\frac{F_0}{c} + g \frac{\rho}{\langle \chi \rangle} \ .$$

The term F_0/c is now a constant and the remaining term is a complicated function of optical depth via its dependences upon the temperature and the density of the material. It's helpful at this point to replace F_0 by its dependence upon the asymptotic temperature, which is the single control parameter regarding the radiation field at our disposal:

$$\frac{dp}{d\tau} = g \frac{\rho}{\langle \chi \rangle} - \frac{4\sigma_R T_\infty^4}{\sqrt{3} c} \ .$$

In Scene 2 we derived four different approximations for $\langle \chi \rangle / \rho$ valid over different temperature ranges. *Each* of the four can be expressed as

$$\frac{\rho}{\langle \chi \rangle} = \frac{\alpha + \beta \rho^\nu T^\mu}{1 + \gamma T^\sigma} = \frac{\alpha + \beta_\star p^\nu T^{\mu - \nu}}{1 + \gamma T^\sigma}$$

for some suitable choice of constants $\alpha, \beta, \gamma, \mu, \nu$ and σ. In the last expression on the right we replace ρ by p and T using the equation of state, and have rescaled $\beta \to \beta_\star$. Our force balance equation now takes the form

$$\frac{dp}{d\tau} = A(T) + B(T) p^\nu = a(\tau) + b(\tau) p^\nu \ ,$$

$$A(T) = \frac{\alpha g}{1 + \gamma T^\sigma} - \frac{4\sigma_R T_\infty^4}{\sqrt{3} c} \ , \qquad B(T) = \frac{\beta_\star g T^{\mu - \nu}}{1 + \gamma T^\sigma} \ ,$$

where A, B, a, b are known functions of their arguments. A sensible static atmosphere in radiative equilibrium is one in which $dp/d\tau > 0$ everywhere. The $B(T)$ term is positive-definite, but the $A(T)$ term can go negative. If $3000K < T < 10^{7.5}K$ our opacity fits select $\alpha = 0$ and $A(T) < 0$.

For the coolest temperatures, from 1500 K to 3000 K where molecules provide the opacity, $\gamma = \beta = B = b = 0$ and

$$\frac{dp}{d\tau} = \alpha g - \frac{F_0}{c} \ , \implies \ p(\tau) = \left(\alpha g - \frac{F_0}{c}\right)\tau \ .$$

At the highest temperatures above several million degrees K, where Thomson Scattering dominates, the exponent $\nu = 1$ and so we have a first-order linear equation, which is also solvable in terms of quadratures. In between, these two extremes, a numerical integration is necessary. For Kramers, $\nu = -1$, and the equation is equivalent to *Abel's Equation of the Second Kind.* And for H^-, $\nu = -1/2$. With $p(\tau)$, $\rho(\tau)$ and $T(\tau)$ now in hand (to varying degrees), the final step is to assign an altitude z to a given τ by integrating

$$\frac{d\tau}{dz} = \langle\chi\rangle(\tau) \ , \qquad z = \int_{\tau}^{\tau_0} dt \frac{1}{\langle\chi\rangle(t)} \ .$$

So much for the lower portions of a stellar atmosphere. How about the Earth's atmosphere? As you spend a considerable fraction of your time living at the base of this atmosphere, you can appreciate some of the subtleties here. At visible wavelengths, the equivalent τ_0 is very very small and the source function is entirely due to scattering during the daytime. This can be corroborated by walking out into the dark at night to see how much the Planck Function contributes to the visible light. At ultraviolet and most infrared wavelengths, τ_0 is now very much greater than unity. In the former case the Planck Function is again negligible (it is all scattered sunlight), while in the latter case the Planck Function is the principal source of photons. The Earth's surface also plays a significant role at visible and infrared wavelengths.

Of course, if we are after the *exact* solution, then we have to retain the Planck Function at all wavelengths, even though its contributions may be minimal. This realization, when translated to the equivalent stellar atmosphere problem (where $\tau_0 \gg 1$) in part helps to motivate why the Hopf Function has such delicate movement between $\tau = 0$ and $\tau \gg 1$—it is taking account of the inconsequential contributions of the source function at very small optical depths in the atmosphere! For visible light in the Earth's atmosphere, it is *all* Hopf Function, so to speak, as far as thermal emission from the atmosphere is concerned. So if you want to be exact, the radiative equilibrium solution is a complete disaster. But if you want to be practical, then it is fairly straightforward—except in the immediate vicinity of $\mu = 0$—can you explain why?

We'll leave it to you to explore these separate cases in greater detail, with two final remarks. First, there is a narrow window around 10 microns where τ_0 plummets from huge to tiny values, and this is how the Earth manages to cool itself. Both carbon dioxide and methane impinge on this window and so the crux of human impacts on climate change hinges upon this radiative transfer problem! Second, Rosseland mean opacities are pointless for the Earth's atmosphere.

4. Planar Geometry. Part 2

Let's add the electromagnetic fields into the mix. If you spend some time musing over the time-independent Maxwell's Equations supplemented by Ohm's Law, you eventually come around to appreciate that barring some really contrived situation (and we will in fact discuss such a contrivance later in this section) we have to insist that $\sigma \to \infty$ and $\mathbf{E} \to 0$ if we also require that $\mathbf{u} = 0$. Finite values of σ invariably convert electromagnetic energy to internal thermal energy of the material and this is prohibited by $\partial/\partial t \equiv 0$.

The energy equation is therefore unaffected. The force-balance equation then becomes

$$\frac{dp}{dz} = -\rho g + \frac{\langle\chi\rangle}{c} F_0 + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \ ,$$

supplemented by the last Maxwell Equation,

$$\nabla \cdot \mathbf{B} = 0 \ ,$$

left standing! In keeping with our planar geometry philosophy, which is that $0 \approx \partial/\partial x, \partial/\partial y \ll \partial/\partial z$, we require that

$$\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = L(z)\hat{\mathbf{e}}_z \ ,$$

for some $L(z)$. If $L(z) \neq 0$, then both $\mathbf{B}$ and $\nabla \times \mathbf{B}$ can have no component in the vertical ($\hat{\mathbf{e}}_z$) direction. Finally, recall that the electric current density is

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \ .$$

Let's dispense with the $L(z) \equiv 0$ case first. Since the magnetic field now has no effect on the energy *and* the force-balance equation, clearly everything we did in the previous section remains valid here. The most general solution is the so-called *force-free field*:

$$\nabla \times \mathbf{B} = \alpha(\mathbf{x})\mathbf{B} \ , \qquad \mathbf{B} \cdot \nabla\alpha = 0 \ .$$

for some function $\alpha(\mathbf{x})$. The second equation tells us that this function α must be constant along a magnetic field line. The choice $\alpha \equiv 0$ provides the *potential magnetic field*,

$$\mathbf{B} = -\nabla\psi \ , \qquad \nabla^2\psi = 0 \ .$$

This magnetic field can have arbitrarily-sized horizontal variations, but, all the other thermodynamic variables are plane-parallel. The choice $\alpha = \alpha_0$, a non-zero constant, corresponds to the so-called *constant-α force-free field*. The remaining possibilities are called *non-linear force-free fields*. The mathematical difficulties encountered in obtaining these fields increase dramatically with the progression from α equals zero, to constant, to full function of position.

For $L(z) \neq 0$, we can take

$$\mathbf{B}(z) = B_1(z)\hat{\mathbf{e}}_x + B_2(z)\hat{\mathbf{e}}_y \ ,$$

for arbitrary functions $B_1(z)$ annd $B_2(z)$, which gives

$$\frac{dp}{dz} = -\rho g + \frac{\langle \chi \rangle}{c} F_0 - \frac{1}{8\pi} \nabla |\mathbf{B}|^2 \ .$$

The third term on the right side of this equation is equivalent to a magnetic pressure. It can obviously have either sign, and therefore it can help to support the material against the downward pull of gravity or it can push down on the material like an additional pseudo-gravitational force. Given our discussion of the tensor virial equations in Act II, in the former case, the material weighs the magnetic field down and prevents it from wanting to expand and escape off to $z \to \infty$. In so far as the two $B_i(z)$ are arbitrary, the *direction* of the magnetic field can change with altitude however one desires.

As there is a vast and rapidly expanding literature devoted to the construction of these magnetostatic atmospheres, I will give you the references where you can find out more, and I will instead end this section with a pedagogical discussion.

Up to this point, we have found RMHD equilibria (in planar geometry) which satsify *both* the energy *and* the force-balance equations. For a variety of reasons, some historical and some simply a matter of expedience, much effort has gone into MHD (notice the absence of the "R") equilibria where the energy equation is largely—with a few extraordinary exceptions—ignored completely. The force-balance equation is

$$\nabla p = -\rho \nabla \Phi + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} \ .$$

When there is an ignorable coordinate, say x_1, with $\partial / \partial x_1 = 0$ there are a variety of means available to find analytic solutions to the resulting equations.

When there are no ignorable coordinates—which is typically the case—any solution must have the property that in the two spatial directions perpendicular to $\nabla\Phi$, the Lorentz Force must be the gradient of a scalar function (i.e., the pressure). This places an additional constraint on the allowable magnetic fields above and beyond $\nabla \cdot \mathbf{B} = 0$. Notice, of course, that since we have tossed the energy equation out the window, why not simply replace the *scalar* gas pressure by a *tensor* and remove this additional constraint. That is one avenue.

Another is to ask how likely is it that all reasonable magnetic fields will satisfy this constraint anyway. The answer is a resounding *not very*. For example, as Boon-Chye Low has demonstrated

$$\nabla \times \mathbf{B} = \nabla \Phi \times \nabla \psi + \alpha(\mathbf{x}) \mathbf{B} \ ,$$

does the trick *provided* $\psi = \psi(\Phi, \mathbf{B} \cdot \nabla \Phi)$. In the present circumstances $\mathbf{B} \cdot \Phi = gB_z$. I know of only one other ansatz that achieves this outcome, again due to Low,

$$\mathbf{A} = A(\mathbf{x}) \hat{\mathbf{e}}_\perp \ , \qquad \hat{\mathbf{e}}_\perp \cdot \nabla \Phi = 0 \ .$$

Hence if it is cosmically unlikely that astrophysical $\mathbf{B}$ arrange themselves in this fashion, it is equally as unlikely that there are accessible MHD equilibria even without worrying about conserving energy! As Eugene Parker has concluded, magnetic *non*-equilibrium is probably the astrophysical norm!

Our second example involves the Earth's global electric circuit and is described in great detail by Feynman in his lectures. It begins by noting that on fair-weather days, there is a prevailing vertical electric field of approximately 100 V/m (sigh, SI units) pointing downward at the base of our atmosphere. The electrical conductivity $\sigma(z)$ is a very strong function of altitude. At ground level, $\sigma(0) \approx 10^{-14}$ Siemens per meter (the Siemen by the way is the inverse of an Ohm, so should these two gentlemen have ever met during their lifetimes their product would have been quite unremarkable), and it increases by 10 orders of magnitudes by $z_1 = 100$ km. The total potential difference over these 100 km is about 400,000 Volts.

There is a constant electric current density (which flows downward)

$$J_0 = \sigma(z)E(z) = -10^{-12}\text{Amps m}^2 \ .$$

As the radius of the Earth is $R_\oplus = 6.378... \times 10^6$ m, we have a total of 511 Amps flowing through the atmosphere into the ground, and the total resistance between the ground and z_1 is 780 Ohms:

$$780 \text{ Ohms} = \int_0^{z_1} \frac{dz}{\sigma(z)} \ .$$

So, our atmosphere is a giant leaky capacitor! Of course, what keeps this whole thing from discharging and being a time-dependent problem are thunderstorms, particularly in the tropics, which although sporadic and intermittent, *on average* drive an equal and opposite net current of 511 Amps in the other direction through lightning discharges! As Feynman points out, since the South Pacific is *the* most popular place for thunderstorms during the late afternoon, there is a daily variation in the fair weather current, and therefore the ground level electric field, on the order of ≈ 10 %.

We can do a little more than Feynman. Since the electric field is

$$\mathbf{E} = \frac{J_0}{\sigma(z)}\hat{\mathbf{e}}_z$$

there must also be an associated magnetic field since

$$c\nabla \times \mathbf{B} = 4\pi J_0 \hat{\mathbf{e}}_z = c\left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2}\right)\hat{\mathbf{e}}_z \ .$$

And the electric charge density is

$$\delta(z) = -\frac{J_0}{4\pi\sigma^2(z)}\frac{d\sigma}{dz} \ .$$

So, the force-balance equation is

$$0 = \left[-\frac{dp}{dz} - \rho g + \frac{J_0^2}{8\pi} \frac{d}{dz} \left(\frac{1}{\sigma^2(z)} \right) \right] \hat{\mathbf{e}}_z + \frac{1}{8\pi} \nabla \left(|\mathbf{B}|^2 \right) \ .$$

The last term on the left is problematic in that it is perpendicular to gravity. But on the other hand, it does carry a $1/c^2$ factor so how important can this be in practice? From a force-balance perspective, not very, but from an energy equation perspective, mission-critical, because

$$\nabla \cdot \mathbf{S} = \nabla \cdot \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) = -\mathbf{J} \cdot \mathbf{E} = -\frac{J_0^2}{\sigma(z)} \ ,$$

and the Poynting Flux has a factor of c in the numerator. So our energy equation reads:

$$\frac{d}{dz} \left(\kappa(p, T) \frac{dT}{dz} \right) = \frac{J_0^2}{\sigma(z)} \ ,$$

where

$$\kappa = \alpha \frac{p}{T} + \beta T + \gamma \ .$$

In the *Exercises* I provide you with actual values for my rough fits for α, β, γ, but the bottom line is that the 511 Amps flowing through the *entire* atmosphere of the Earth—which by the way has a mass of about 5×10^{18} kg—the Joule heating on the right side of this equation is absolutely pitiful compared to anything else we might have overlooked.

5. Spherical Geometry

Now we have

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r) \ ,$$

which adds a fair degree more complication to everything than one might have guessed ab initio. On the other hand, the substitution

$$r = \frac{1}{\zeta}$$

has the salubrious effect of *almost* turning the spherical geometry equations into the planar geometry equations with the association $z \leftrightarrow \zeta$. To wit,

$$\frac{d^2\Phi}{d\zeta^2} = \frac{4\pi G}{\zeta^4} \rho(\zeta^{-1}) \ \ \text{versus} \ \ \frac{d^2\Phi}{dz^2} = 4\pi G \rho(z) \ ,$$

the energy equation is

$$\kappa \frac{dT}{d\zeta} + \frac{1}{\zeta^2} F = \text{constant} \ \ \text{versus} \ \ -\kappa \frac{dT}{dz} + F = \text{constant} \ ,$$

and for force-balance we have (either way)

$$\frac{dp}{d\Phi} + \rho = \frac{\langle \chi \rangle}{c |\nabla \Phi|} F \ .$$

Indeed, if we omit the radiation field and self-gravity, the correspondence is *exact*! Exact, except for one minor, and as it turns out, terribly important, detail. The outer $r \to \infty$ part of our atmosphere is now in the neighborhood of $\zeta = 0$, and the base of our atmosphere is someplace way out in the vicinity of $\zeta \gg 1$.

All the solutions we derived in §3 for thermal conduction and externally-supplied gravity can be brought over with the simple replacement $z \to 1/r$. However, fitting boundary conditions now turns out to be a serious headache. In particular, it is very hard to avoid having a finite pressure as $r \to \infty$. The only situation in which this can be achieved is for air in the Earth's atmosphere, but admittedly this involves assuming my oddball fit to the thermal conductivity works in the limit of zero temperature, which it decidedly does not! Precisely such issues are what motivated Parker to think about dropping the $\mathbf{u} = 0$ assumption and looking for steady ($\partial/\partial t \equiv 0$) winds from stars.

In any case, we now get to work with self-gravity in a meaningful way! Perhaps the simplest of all possible scenarios is an isothermal self-gravitating sphere. The energy equation is trivially satisfied. The force-balance equation integrates to give

$$p = p_0 \exp\left(-\frac{\Phi}{(c_p - c_v)T_0}\right)$$

in terms of two integration constants, T_0 and p_0. The Poisson Equation is now

$$\frac{d^2\Phi}{d\zeta^2} = -\frac{4\pi G}{(c_p - c_V)T_0}\zeta^{-4} \exp\left(-\frac{\Phi}{(c_p - c_v)T_0}\right) \ ,$$

or

$$\xi^4 \frac{d^2\Psi}{d\xi^2} + e^{\Psi} = 0$$

in dimensionless form. And that is the end of the line as far as analytic results are concerned. Period. Game over.

The (numerically generated) solutions to this equation are called *Bonnor-Ebert Spheres*. For the Bonnor-Ebert Sphere, the density can be crudely fit by

$$\rho(r) \approx \frac{\rho_0}{1 + \alpha r^2}$$

and so both the density and the pressure fall-off as r^{-2}. Here, the central density ρ_0, like the temperature T_0, is a control parameter and $\alpha(\rho_0, T_0)$ is determined by fitting the numerical solution. The gravitational potential grows with r like $\log r$, consistent with the fact that as $r \to \infty$, the Bonnor-Ebert Sphere contains an infinite amount of mass! For this reason we do not have the issue with a finite pressure at infinity that we encountered above. There is however an enclosed mass beyond which the Bonnor-Ebert Sphere is unstable to gravitational collapse called, unimaginatively, the *Bonnor-Ebert Mass*:

$$M_{BE} = 3.77 M_\odot \left(\frac{2.3}{\mu}\right)^{3/2} \left(\frac{T_0}{1 \text{ deg K}}\right)^{3/2} \left(\frac{N_0}{1 \text{ cm}^{-3}}\right)^{-1/2} \ ,$$

where μ is the mean molecular weight of the gas, and N_0 is the number density at $r = 0$.

By "unstable" we have in mind the following. Suppose we have a Bonnor-Ebert Sphere with central density ρ_0 and uniform temperature T_0. At some radius r, we shrink the interior just slightly by an amount dr and ask for the unique Bonner-Ebert Sphere which encloses the same amount of mass at the same temperature T_0 but now at a central density which is slightly greater by an amount $d\rho_0$. This solution will have a slightly different density/pressure at the radius r where we have enforced the slight shrinkage. If the change in pressure is positive then the sphere is stable (it pushes back at us when we try to compress it), if the change is negative then we say it is unstable.

So much for self-gravity. Everything else we may wish to do here must eventually come down at least to the numerical integration of a nonlinear ODE. This to my way of thinking is fairly disappointing. On the other hand, it probably has to end up this way because, ultimately, we are actually building entire stars with these equations (albeit spherically-symmetric, static and non-rotating ones, to be sure) and how easy should that actually be?

Rarely, the mathematics sometimes is kind to us. If we go back to

$$\frac{dp}{d\Phi} + \rho = 0 \ ,$$

and instead of Bonnor-Ebert with $p \propto \rho$ we take $p \propto \rho^{1+1/n}$, several things happen. First, of course, we do *serious* damage to the energy equation, which we shall have to patch up somehow! Second, the pressure, density and temperature are now powers of the gravitational potential

$$p = p_0 \Phi^{n+1} \ , \qquad \rho = \rho_0 \Phi^n \ , \qquad T = T_0 \Phi \ .$$

Third, in dimensionless form, the Poisson Equation is now the *Lane-Emden Equation*:

$$\xi^4 \frac{d^2\Psi}{d\xi^2} + \Psi^n = 0 \ .$$

Although this looks for all intents and purposes as bad as the Bonnor-Ebert result, should we be fortunate enough to select $n = 0, 1, 5$ then this equation can be integrated *exactly*. Bonnor-Ebert in some sense corresponds to the limit $n \to \infty$. Just for the record, the solutions are

$$\Psi_0 = 1 - \frac{1}{6\xi^2} \ ,$$

$$\Psi_1 = \xi \sin \frac{1}{\xi} \ ,$$

$$\Psi_5 = \left(1 + \frac{1}{3\xi^2}\right)^{-1/2} \ .$$

A particularly fascinating aspect of nonlinear ODEs is that *uniqueness* of solutions is not a given. For example Ψ_5 has a bizarre sibling (better a half-brother or half-sister):

$$\Psi_{\text{“}5\text{”}} = \frac{\xi^{1/2}\sin[\log \xi^{-1/2}]}{3 - 2\sin^2[\log \xi^{-1/2}]} \ ,$$

which does not fit the bill because of its outrageous behavior, particularly as $\xi \to \infty$ or $r \to 0$.

It remains to say something, albeit very little in fact, about radiative transfer in spherical geometry. The Appendix D of Act I Scene 4 actually goes a long way in setting the stage and indicating the general lines along which such an endeavor must proceed. I will not bother to belabor those points again here.

6. Summary

We have used the tools we developed in Acts I and II to investigate the behavior of various types of equilibria where radiation, gravity and electromagnetic fields all play a role. We only scratched the surface. Hopefully in reading through this Scene you paused several times to imagine ways in which you could improve upon the treatment and develop a more realistic equilibrium state. Have at it! Don't suppress your imagination.

Once you have an equilibrium of any sort, it is useful to ask if it is stable. We raised the issue in passing in our discussion of the Bonnor-Ebert Spheres. Much more can be said about stability analyses. Unstable equilibria are fairly useless in the sense that they can never be realized for any sensible extent of time and so the universe does not waste much effort on them. Neither should we, except in so far as they may be of pedagogical value. Stable equilibria often support a variety of small-amplitude disturbances collectively referred to as *waves* or *oscillations.*

The next step is to look for RMHD equilibria that possess steady flows!

7. Exercises

Exercise 1: <u>WHY IS IT −55 C AT 30,000 FEET?</u>
If you do much flying at all, you are well aware that the temperature drops rapidly with altitude and that it is pretty chilly outside at cruising flight levels. Does the solution of the thermal conduction problem for our plane-parallel atmosphere get the temperature right at 30,000 feet?
(A) For the constants α, β, γ in the expression

$$\kappa = \alpha\frac{p}{T} + \beta T + \gamma$$

I used a website called *The Engineering Toolbox.* They plot κ in mW/m/deg K (that's milliWatts per meter per degree Kelvin) versus temperature in degrees Celsius between −200 and +1600, for a variety of atmospheric pressures ranging from 1 bar to 1000 bars. Don't you just *love* these units? Here is the site

https://www.engineeringtoolbox.com/air-properties-viscosity-conductivity-heat-capacity-d_1509.html and the plot I am using is the third down from the top of the page. (The same information is in the fourth plot as well but now in British Thermal Units per hour per foot per degree Farenheit—ouch!) For T in degrees Kelvin, p in units of 1000 bar (which is almost what the pressure is at ground level), and κ in mW/m/deg K, I get

$$\alpha \approx 16600 \ , \qquad \beta \approx \frac{1}{20} \ , \qquad \gamma \approx 44 \ .$$

What do you estimate these constant to be? How does the linear dependence on pressure look to you?
(B) Starting out with a reasonable ground temperature and pressure, integrate your way up to 30,000 feet and see what you get. If things seem slightly awry look at (C) below.
(C) Somewhere between 30,000 and 50,000 feet altitude, depending upon your latitude, the temperature levels off and subsequently starts to increase with altitude. Why might that be?

Exercise 2: BONNER-EBERT WITH THERMAL CONDUCTION
If you have access to a numerical integration routine you can build your own Spheres (name them after yourself or your favorite pet if you like) with thermal conduction.
(A) First verify that the solution of the thermal conduction equation in spherical geometry is

$$T = \left(\frac{F_0}{r} + T_\infty^{7/2} \right)^{2/7} \ ,$$

with two integration constants. One of these, F_0, accounts for the outward energy flux. Notice that at large r, Your Sphere is going to behave just like the Bonner-Ebert Sphere! Your Sphere is unfortunately not going to be very well-behaved as $r \to 0$ unless we patch things up there. To do this postulate that there is an energy source that is a non-zero constant over a tiny innermost sphere of radius $0 \leq r \leq R_0$. The divergence of the conductive flux is equal to this constant (not zero) in this innermost energy-generating core.
(B) Convince yourself that in this inner energy-generating core, the temperature must behave like

$$T = \left(T_0^{7/2} - F_1 r^2 \right)^{2/7} \ ,$$

with two more integration constants. One is the temperature at the core of the sphere. The other depends upon the amount of energy you are generating. By patching this solution onto the one you found in part (A) at $r = R_0$ you can find a relationship between the energy generated inside this core and the conductive energy flux that emerges as $r \to \infty$ outside of this core.
(C) Now use the force-balance equation and the equation of state to obtain

$$\frac{d\Phi}{dr} = -\frac{(c_p - c_V)T(r)}{p(r)} \frac{dp}{dr} \ ,$$

and substitute this into Poisson's Equation to obtain an equation for $p(r)$.

(D) If you have access to a numerical integration routine, then apply it to the equation you derived in part (C) to obtain your own Spheres!

Exercise 3: FINDING A HOTTER PLACE

Consider the plane-parallel radiative equilibrium where the flux F_0 or equivalently the asymptotic temperature T_∞ is so large that you can use the fit to the Thomson Scattering opacity

$$\frac{\rho}{\langle\chi\rangle} = \frac{\alpha + \beta\rho^\nu T^\mu}{1 + \gamma T^\sigma} = \frac{\alpha + \beta_\star p^\nu T^{\mu-\nu}}{1 + \gamma T^\sigma}$$

everywhere in the atmosphere. Although all of these constant are non-zero, the saving grace is that $\nu = 1$ and your force-balance equation

$$\frac{dp}{d\tau} = A(T) + B(T)p = a(\tau) + b(\tau)p \ ,$$

can be integrated exactly.

(A) How large can F_0 or T_∞ be taken before $dp/d\tau$ goes negative somewhere in the atmosphere for a specified value of the surface gravity, g? What does this imply?

Exercise 4: DR. SIEMENS I PRESUME?

Take the exercise we carried out at the end of §4, and place it in the proper spherical geometry. This will help if you were concerned that it was not possible to decide how much B_1 versus B_2 we should take. The answer depends on just where you are, of course, but in Montreal it would be mostly in the East-West direction and increasing as you move southward.

(A) What sort of magnetic field is generated in a spherical shell due to a radial current that varies as r^{-2} and how does this compare with the actual poloidal magnetic field of a few Gauss generated by the Earth's geomagnetic dynamo? Remember that the *total net current* between the two concentric spherical shells is actually zero—the 511 Amps downward distributed uniformly over the globe has to be balanced somewhere by the return 511 Amp current due to the thunderstorms. You could start by distributing this return current uniformly around the equator and then concentrate it more in the South Pacific if you want to get really clever. So what direction does the field *actually* point in Montreal?

(B) On average there are 50 lightning strikes per second over the entire Earth. Their overall average duration is about 0.2 seconds but they consist of numerous intense bolts that last only for tens of microseconds. The peak power during a lightning strike is about 10^{12} Watts and currents can be several hundred thousand Amps. How does this square with the 511 Amp return current required to close the global electric circuit? [Hint: What is the altitude of the a typical thunderstorm cloud top?] If you get stuck, you might look up something about "sprites".

(C) Assuming the top and bottom of the spherical atmospheric shell are fairly good conductors, estimate the amount of electromagnetic radiation produced by a lightning stroke. [Hint: Go back to Act I Scene 3.] If you get stuck page through Rakov & Uman [**RU 1**] and look up the word "sferics".

(D) How likely *is* it that Ernest Werner von Siemens and Georg Simon Ohm actually did meet in person? Can you think of two other scientists who, when multiplied upon meeting, would give unity? What would happen if Sir Isaac Newton stood within a square a little over three fieet on a side? [Hint: How many bars—or fraction of a standard atmopshere—would he represent in so doing?]

8. Further Reading

The variation of electrical conductivity with altitude, and numerous other fascinating things about the Earth's global electric circuit can be found in
⋆[**V 2**] Hans Volland, Atmospheric Electrodynamics, (Berlin, DE: Springer-Verlag; 1984), ix+205.
The discussion by Feynman is in
⋆[**FLS 1**] Richard P. Feynman, Robert B. Leighton & Matthew Sands, The Feynman Lectures on Physics. Volume II. Mainly Electromagnetism and Matter, (Reading, MA: Addison-Wesley Publishing Company; 1975).
Finally, because it is germane and is such an amazing compilation of information on the topic, have a look at
[**RU 1**] Vladimir A. Rakov & Martin A. Uman, Lightning. Physics and Effects, (Cambridge, UK: Cambridge University Press; 2003), x+687.

Boon-Chye Low is without doubt one of the most brilliant MHD theorists of his, and for that matter, probably any other generation. He has pushed the boundaries of analytic theory far beyond what was imagined to be possible. The two fully three-dimensional MHD solutions described above are derived in
[**L 6**] B.C. Low, "Magnetostatic atmospheres with variations in three dimensions", *Astrophysical Journal*, **263**, 952-69, 1982,
[**L 7**] B.C. Low, "Three-dimensional structures of magnetostatic atmospheres. III. A general formulation", *Astrophysical Journal*, **370**, 427-34, 1991,
[**N 1**] T. Neukirch, "On self-consistent three-dimensional analytic solutions of the magnetohydrostatic equations", *Astronomy & Astrophysics*, **301**, 628-39, 1995.
The numerous earlier references on MHD will provide lots of background on various magnetostatic equilibria in a variety of geometries. For a careful explanation of Parker's thesis on the general lack of magnetostatic equilibria in nature, dig into
[**P 9**] Eugene N. Parker, Spontaneous Current Sheets in Magnetic Fields. With Applications to Stellar X-Rays, (New York, NY: Oxford University Press; 1994), xiv+420.

An in-depth treatment of Bonnor-Ebert Spheres and solutions of the Lane-Emden Equation can be obtained from
⋆[**C 9**] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, (New York, NY: Dover Publications; 1957), iii+508,
with no particular assistance from J.B. Sykes that I am aware of.

Radiative transfer (and thermal conduction) in the Earth's atmosphere is a good deal more complex and exciting than I have made it out to be here. To

gain a sense of just how much more effort can go into this subject, have a glance at Goody [**G 4**],
[**K 5**] K. Ya. Kondratyev, Radiation in the Atmosphere, (New York, NY: Academic Press; 1975), xvi+912,
[**R 6**] Georgii Vladimirovich Rozenberg, Twilight. A Study in Atmospheric Optics, (New York, NY: Plenum Press; 1965), x+358 ,
[**H 8**] Bruce Hapke, Theory of Reflectance and Emittance Spectroscopy, (Cambridge, UK: Cambridge University Press; 1993), xiii+455.
Indeed, what I have skipped over and which adds immeasurably to the complexity of the issues, is the presence to varying degrees of water vapor, and its dramatic manifestation as clouds. This is a complicated subject no matter how one approaches it. A very nice book that complements the general development of this *Opera* is:
[**DD 1**] Louis Dufour & Raymond Defay, Thermodynamics of Clouds, (New York, NY: Academic Press; 1963), xiii+255.

Waves and oscillations come in both linear and nonlinear varieties and reflect (no pun intended) their restoring forces, be they electromagnetic, gravitational, compressibility, rotational and so forth, which support them. There is no shortage of books on this subject. The following selection attempts to cover the range of topics we have dealt with while at the same time maintaining elegance and clarity of exposition.
[**UOASS 1**] Wasaburo Unno, Yoji Osaki, Hiroyasu Ando, Hideyuki Saio & Hiromoto Shibihashi, Nonradial Oscillations of Stars, 2nd Edn, (Tokyo, JP: University of Tokyo Press; 1989,
[**C 10**] Alex D.D. Craik, Wave Interactions and Fluid Flows, (Cambridge, UK: Cambridge University Press; 1990), xii+322,
[**P 10**] Joseph Pedlosky, Waves in the Ocean and Atmosphere. An Introduction to Wave Dynamics, (Berlin, DE: Springer; 2003), viii+260,
[**S 10**] Thomas Howard Stix, The Theory of Plasma Waves, (New York, NY: McGraw-Hill Book Company; 1962), x+283,
[**B 9**] Tom Beer, Atmospheric Waves, (London, UK: Adam Hilger; 1975), xii+300,
[**B 10**] D.I. Blokhintsev, Acoustics of a Nonhomogeneous Moving Media, (Washington, DC: National Advisory Committee for Aeronautics; 1956), Technical Memorandum 1399, iv+194, https://ntrs.nasa.gov/search.jsp?R=19930091111
[**L 8**] James Lighthill, Waves in Fluids, (Cambridge, UK: Cambridge University Press; 1980), xv+504,
[**W 3**] G.B. Whitham, Linear and Nonlinear Waves, (New York, NY: John Wiley & Sons; 1974), xvi+636.

How does one know that something is "Abel's Equation of the First Kind" and therefore one should not waste effort in looking for integrating factors? How do we know that the Lane-Emden Equation has three (and a half-sibling) analytic solutions, while the Bonnor-Ebert Equation has none? The answer is that we rely on the knowledge of people who have been there and done that and tabulated their findings. For example,

[**PZ 1**] Andrei D. Polyanin & Valentin F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, (Boca Raton, FL: CRC Press; 1995), xv+707,
[**S 11**] P.L. Sachdev, A Compendium on Nonlinear Ordinary Differential Equations, (New York, NY: John Wiley & Sons; 1997), xi+918,
[**K 6**] E. Kamke, Differentialgleichungen. Lösungsmethoden und Lösungen. I. Gewöhnliche Differentialgleichungen II. Partielle Differentialgleichungen Erster Ordung für Eine Gesuchte Funktion, 10th Auflage/6th Auflage, (Stuttgart, DE: B.G. Teubner; 1983), xxvi+668/xiii+243 .
While you are at it, you may as well add the following to your reference library, because it has saved me too many times to count
[**H 9**] Eldon R. Hansen, A Table of Series and Products, (Englewood Cliffs, NJ: Prentice-Hall; 1975), xviii+523.