Radiation Magnetohydrodynamics: An Opera in Three Acts

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1. All the Universe's is a Stage,

The drama of radiation magnetohydrodynamics, herein RMHD, and the diverse astrophysical systems it describes takes place on the stage of a 3+1 dimensional space-time. We say, three-plus-one, four total, because the spatial dimensions–up/down, north/south, east/west—have much in common while time is something altogether different. This assertion is based in large part on our everyday experiences, and experiment. As Hermann Weyl once famously remarked, in odd-dimensional space-times, *odd* things can happen, like rooms do not go dark immediately after one blows out a candle. We'll take it as a given that our space-time dimension, and the stage upon which the drama of RMHD unfolds, is an *even* 4 dimensions, so that candles do the honorable thing when extinguished.

Much more can be said about the stage, the scenery and the lighting. Mathematically, one usually takes space-time to be a *vector space* over the scalar *algebraic field* of real numbers, denoted by $\mathbb{R}$. These are fairly technical but fundamentally important characterizations of the stage and scenery. We'll say more about precisely what this means later. But basically a vector space is a (admittedly very large) set of *elements*—also called *vectors*—which can be added and multiplied by scalars to yield other vectors which live in the vector space. One can also profitably think about the elements of our 3+1 space-time vector space as *events* described by a (three-dimensional) location and (one-dimensional) time.

A *precise* definition of our space-time stage requires that one specify the set (or symmetry group) of linear transformations (or automorphisms) from the vector space to itself which leave the equations of RMHD invariant. Another way to think of this is to accept that observers all over the universe would like to set up their coordinate systems and clocks independently of one another. All the allowable setups in which the equations of RMHD look the same are connected by this special set of linear symmetry transformations. The set of linear symmetry transformations forms an algebraic structure called a *Lie Group*, which is simpler than an *algebraic field*.

Newton's laws of motion and gravitation, and all the classical physics they describe, are invariant under the *Galilean Group*. The Galilean space-time is one in which observers all use one universal clock, agree on simultaneity of events in space-time, and agree on accelerations of objects (but *not* their velocities or positions). This seemed to be a pretty good 3+1 space-time until Maxwell finished putting the displacement current in the electromagnetic field equations, which then implied that light was an electromagnetic wave which propagates at the same velocity c in any setup in which Maxwell's Equations are valid.

Every observer in the universe would like to use the *same* set of Maxwell

Equations to describe their electromagnetic fields rather than refer them all to some universal coordinate system based lord knows where and oriented somehow. The Galilean Group does not permit this. So it was necessary to come up with a new symmetry group, the *Poincaré Group*, and a corresponding *Minkowski* space-time which allows all observers (in sensible frames of reference!) to use the same Maxwell's Equations. In Minkowski space-time, observers no longer agree on simultaneity, there is no universal clock, and they don't agree on accelerations either. Luckily, for observers that are moving very slowly (i.e., much less than the speed of light) relative to one another, the transformations of the Poincaré Group look very much like the transformations of the Galilean Group, and so to a high degree of approximation, they can keep both Newton and Maxwell (almost) invariant!

Additional experiments now suggest that our 3+1 space-time is a Minkowski space-time only *locally*. Minkowski space-time is in the parlance of general relativity, a *flat* space-time. The actual space-time in which we live is now thought to be locally flat, but is globally curved in a Riemannian sense by the presence of matter and energy. The curvature of space-time is de facto the generalization of Newton's theory of gravitation in the hands of Einstein.

2. And All Matter and Fields Merely Players;

The players in our RMHD drama are the material, or matter, and the electromagnetic, radiation and gravitational fields. And they all play their parts on the 3+1 dimensional space-time stage. The concept of a *physical* (as opposed to an *algebraic*) field originated with Faraday. It is at the same time a very elegant, and extremely confusing, concept, because of the underlying premise of action-at-a-distance. The philosophical debates about the "reality" of fields are contentious and complex. We leave them to those who enjoy and excel at these sorts of things.

There is a one-to-one correspondence between fields and forces. Of the four fundamental forces in nature, only two are long-range, having quanta with zero mass: electromagnetism (photon) and gravitation (graviton). And therefore only these two have classical incarnations. These are in fact the only two fields which have sensible roles in the RMHD *Opera*.

Like matter, (physical) fields can store momentum, energy and angular momentum. In the unfolding drama of RMHD, energy, momentum and angular momentum are transferred between the material and the two long-range fields. One would like to understand how, why, when and how much exchange takes place in order to quantify the dynamics and predict the behavior of astrophysical systems. We'll concentrate on energy and momentum in what follows. You can add in angular momentum as you see fit.

Mathematically, the characterization of the material and the fields live in the *tangent space* affixed to each event in the 3+1 dimensional (vector space) space-time. The set of all tangent spaces over a vector space is called a *vector bundle*. This is about all we will try to say about vector bundles.

Although we believe that the matter is ultimately a gas or aggregate of discrete, localized, entitites, referred to loosely as *particles*, our approach is to smooth out their clumpy intermittency (in space-time) in favor of a continuum or fluid description. In fact, from a quantum mechanical perspective, these particles are localized only to a certain degree and exhibit both particulate and wave-like behavior. This adds yet another level of complication to the reality of astrophysical material. In some situations, such as the interiors of white dwarf and neutron stars, the quantum mechanical aspects cannot be avoided.

The continuum fluid description can be derived in a consistent fashion from the statistical mechanics of the material. A particular constituent, like electrons, say, is described by a distribution function defined on a 6-dimensional phase space such that the number of electrons in a small volume element $d\mathbf{x}$ is (superscript "e" for electron)

$$dN^e = N^e(\mathbf{x}, t)d\mathbf{x} = d\mathbf{x} \sum_{\pm 1/2} \int d\mathbf{p} \; f^e_{\pm 1/2}(\mathbf{x}, \mathbf{p}, t) \ ,$$

where the sum is over the two spin states of the electron. Similar expressions obtain for protons and Helium atoms and Iron ions and so forth. Such a description is tenable if the number density of constituents is not too large that their spatial extent begins to limit the available free space between them. When this occurs, two-particle correlations must be taken into account, as in the treatment of liquids and dense gasses.

In the amazing state of grace known as *statistical equilibrium*, the electron distribution takes on a universal form consistent with the electrons being spin-1/2 Fermions:

$$f^e_{\pm 1/2} = \frac{1}{h^3} \; \frac{1}{\exp(\mu_e + \varepsilon_p / k_B T) + 1} \ ,$$

where $h = 6.6261... \times 10^{-27}$ erg sec, is Planck's Constant, and $k_B = 1.3807... \times 10^{-16}$ erg per degree Kelvin, is Boltzmann's Constant. The *chemical potential*, $\mu_e(T)$ is required to ensure number conservation and ε_p is the energy of an electron with momentum $\mathbf{p}$. For large but still subrelativistic electron energies, this essentially becomes the *Maxwellian Distribution*

$$f^e_{\pm 1/2} \approx \frac{1}{h^3} \exp\left(-\mu_e - \frac{|\mathbf{p}|^2}{2 m_e k_B T} \right) \ ,$$

where $m_e = 9.1094... \times 10^{-28}$ gm is the mass of the electron.

Of course what makes astrophysical systems interesting to us is that they are *not* generally speaking in any sort of equilibrium and it is their subsequent evolution, perhaps toward an equilibrium, or perhaps not, which we would like to capture and understand. In the guise of RMHD, we should like to capture this behavior with the minimum investment in following details—like the evolution of phase space distribution functions—but which still permits an accurate assessment of outcomes. What in a great many instances comes to our assistance in this endeavor is that while astrophysical systems are not in statistical equilibirum globally, much less locally, they are often not particularly "far away" in a certain sense from this state of grace. This proximity permits a methodology in-the-large which is unusually accurate (all things considered) in describing the subsequent efforts of the system to relax toward both a local and global statistical equilibrium.

The continuum fluid description of the matter is the method of choice to achieve this outcome when the typical mean-free-paths for particles between collisions are everywhere small compared to dynamical length scales of interest. This need not be true for photons, which can often attain mean-free-paths comparable not only to dynamic length scales, but the entire breadth of the astrophysical system. The "photon gas" cannot be described in the manner of a fluid, but it must be treated in such a fashion that both very large and very small mean free paths can be accommodated as necessary. This is the essential framework of RMHD that we develop here.

It is worth noting in passing that particles of extremely high energy, that is $\varepsilon_p \gg k_B T$, likewise can travel large, possibly macroscopic, distances between collisions with the material, or gyro-resonant scattering from low-frequency electromagnetic waves. As such particles are usually extremely rare, their impacts on bulk material properties can generally be neglected. When they cannot, then a fairly obvious extension of the RMHD formalism used to treat photons can be applied to the high-energy tail of the matter's distribution function, but with the important caveat that particles, unlike photons, are conserved.

3. They Have Their Exits and Entrances;

RMHD is best formulated in terms of a set of *conservation laws*, for quantities that we believe/assert must be conserved by the astrophysical system as it evolves dynamically toward an equilibrium state. The essential conserved quantites are *mass*, the full *momentum* vector (each of the three spatial components individually), and the *energy*. Energy and (vector) momentum can be stored and transported by the material, the gravitational and electromagnetic and radiation fields. The mass resides entirely with the material. It follows that RMHD consists of 17 conservation laws: one for mass and (3+1)×4 for (vector) momentum (3) and energy (1), for the material (1) , gravitational (2), electromagnetic (3), and radiation (4) fields, individually.

Of the latter 16 equations, each is of *conservation* form, for example: the time derivative of the energy density plus the divergence of an energy flux vector, equals the exchange of energy between the matter and the three fields. Symbolically:

$$\frac{\partial E^\alpha}{\partial t} + \nabla \cdot \mathbf{F}^\alpha = \sum_\beta (\dot{\mathcal{E}}^{\beta\to\alpha} - \dot{\mathcal{E}}^{\alpha\to\beta})$$

where α, β run over $M-$"matter", $G-$"gravity", $EM-$"electromagnetic", and $R-$"radiation". Because

$$\dot{\mathcal{E}}^{\alpha\to\beta} + \dot{\mathcal{E}}^{\beta\to\alpha} = 0 \ , \alpha \neq \beta$$

and $\dot{\mathcal{E}}^{\alpha\to\alpha} = 0$ there are 6 independent exchange terms for the energy conservation equation. The *total* energy, irrespective of where it lies, must be conserved:

$$\frac{\partial}{\partial t}\sum_{\alpha} E^{\alpha} + \nabla\cdot\sum_{\alpha}\mathbf{F}^{\alpha} = 0\ .$$

The essential goal of RMHD is therefore to determine the 4 densities, the 4 fluxes and the 6 exchange terms (14 quantities) for each of the 4 conservation of (vector) momentum and energy equations (56 quantities in total). There are no exchange terms for mass conservation—just the mass density ρ and the mass flux $\rho\mathbf{u}$, where $\mathbf{u}$ is the fluid velocity—so the grand total is (count 'em) 58 quantities. Luckily, several are zero.

Finally, for the three fields, the conservation laws must be supplemented by *field equations*. For gravity, this is the Newton-Poisson Equation. For electromagnetism, these are Maxwell's Equations. For the radiation field, it is the transfer equation.

4. And Each in Their Time Plays Several Parts.

The electromagnetic (and perhaps the gravitational?) field is best described in two very different fashions depending upon its temporal variability. At *low frequencies*, quasi-static electromagnetic fields are treated in a deterministic fashion directly from the classical governing field equations, which are Maxwell's Equations. At *high frequencies*, a probabilistic/statistical approach is preferable. For the electromagnetic fields, this high-frequency statistical treatment is referred to as *radiative transfer*. Gravitational waves have only just been detected. Never-the-less in the future one could envision a theory of gravitational radiative transfer which describes the behavior of the gravitational radiation field.

The dividing frequency/wavelength which separates the deterministic and probabilistic treatments of electromagnetism is usually determined on a case-by-case basis, but crudely speaking an interesting milepost in such complicated deliberations is

$$\frac{k_B T}{h} = 2.084\times 10^{10}\left(\frac{T}{1\ \mathrm{deg\ K}}\right)\ \mathrm{sec}^{-1}\ ,$$

$$\frac{hc}{k_B T} = 1.440\times 10^{-1}\left(\frac{1\ \mathrm{deg\ K}}{T}\right)\ \mathrm{cm}\ ,$$

for a radiation/electromagnetic field that is in statistical equilibrium with the matter at a temperature T [dimensions: deg Kelvin]. Here, k_B is Boltzmann's Constant and h is Planck's Constant and $c = 2.9979...\times 10^{10}$ cm sec^{-1}, is the speed of light.

The capacity, and indeed the necessity, to formulate a description of the electromagnetic field concurrently as both a large-scale (low-frequency) wave and as a statistical ensemble of discrete (high-frequency) particles is a manifestation of the wave-particle duality essential to quantum mechanics. Photons are Bosons with spin-one angular momentum $h/2\pi$, but, because they travel at the speed of light (by definition) only two, the $\pm\hbar$, of the three spin angular momentum substates are realized, corresponding to right-hand circular polarization and left-hand circular polarization, respectively.

Let $f^{ph}_{\pm 1}(\mathbf{x}, \mathbf{p}, t)$ denote the (single) photon distribution function for right-hand $(+1)$ and left-hand (-1) circular polarization, so that

$$dN^{ph} = N^{ph}(\mathbf{x}, t)d\mathbf{x} = d\mathbf{x} \sum_{\pm 1} \int d\mathbf{p} \; f^{ph}_{\pm 1}(\mathbf{x}, \mathbf{p}, t)$$

is the number of photons in a volume element $d\mathbf{x}$ at time t. The relationship between the momentum of the photon $\mathbf{p}$ and its direction of propagation $\mathbf{n}$ (a dimensionless unit vector) is

$$\mathbf{p} = \frac{h}{2\pi}\mathbf{k} = \frac{h\nu}{c}\mathbf{n} = \frac{h}{\lambda}\mathbf{n} \; .$$

The fundamental discovery of the late 19th and early 20th century was that if the photons are in statistical equilibrium with the material at a common temperature T, then there is a universal form for this distribution function:

$$f^{ph}_{\pm 1} = \frac{1}{h^3} \; \frac{1}{\exp(h\nu/k_B T) - 1} \; ,$$

irrespective of the sense of circular polarization. (The chemical potential of the photon, unlike the electron, is zero, essentially because their number is not conserved!) As h^3 is the fundamental volume element in phase space, the photon *occupation number* is

$$\mathcal{N}^{ph}_{\pm 1} \equiv \frac{1}{\exp(h\nu/k_B T) - 1}$$

as is appropriate for particles which obey Bose-Einstein statistics. For $h\nu \gg k_B T$, the occupation numbers are tiny and the distribution is Maxwellian in the sense that

$$f^{ph}_{\pm 1} \approx \frac{1}{h^3}\exp(-h\nu/k_B T) = \frac{1}{h^3}\exp(-\varepsilon_\nu/k_B T)$$

where ε_ν is the energy of a photon. However, when $h\nu \ll k_B T$ then the occupation numbers are immense (as is the case for Bosons). Fermions, thanks to the Pauli Exclusion Principle can have only one particle in a unit phase space volume element, hence for Fermi-Dirac Statistics, we must replace the "-1" with a "$+1$" in the denominator for $\mathcal{N}$.

Noting that

$$d\mathbf{p} \equiv dp_1 dp_2 dp_3 = \frac{h^3\nu^2}{c^3} d\nu d\mathbf{n}$$

we can write

$$dN^{ph} = N^{ph}(\mathbf{x}, t)d\mathbf{x} = d\mathbf{x} \sum_{\pm 1} \int_0^\infty d\nu \int d\mathbf{n} \; \frac{h^3\nu^2}{c^3} f_{\pm 1}(\mathbf{x}, t; \mathbf{n}, \nu) \; ,$$

and similarly for the energy

$$dE^{ph} = E^{ph}(\mathbf{x}, t)d\mathbf{x} = d\mathbf{x} \sum_{\pm 1} \int_0^\infty d\nu \int d\mathbf{n} \; \frac{h^4\nu^3}{c^3} f_{\pm 1}(\mathbf{x}, t; \mathbf{n}, \nu) \; .$$

In terms of the *specific intensity* of the radiation field, which we shall discuss at length in Act I Scene 4, $I_\nu(\mathbf{x}, t : \mathbf{n})$, we also have

$$dE^{ph} = I_\nu(\mathbf{x}, t; \mathbf{n})\ [cdt\ \mathbf{n} \cdot d\mathbf{S}]\ d\nu\ d\mathbf{n}\ ,$$

where $d\mathbf{S}$ is the normal to an element of surface of area $|d\mathbf{S}|$. The factor in square brackets is $d\mathbf{x}$. Comparing these two expressions we obtain

$$I_\nu(\mathbf{x}, t; \mathbf{n}) = \sum_{\pm 1} \frac{h^4 \nu^3}{c^2}\ f_{\pm 1}(\mathbf{x}, t; \mathbf{n}, \nu)\ .$$

Therefore in local thermodynamic equilibrium, we have

$$I_\nu(\mathbf{x}, t) = \frac{2h\nu^3}{c^2}\ \frac{1}{\exp(h\nu/k_B T) - 1} \equiv B_\nu[T]$$

where $B_\nu[T]$ is the *Planck Function*, and $T(\mathbf{x}, t)$ is the temperature of the material and the radiation field. (Notice the factor of “2” counts each of the two circular polarization states.

At low frequencies, where we shall work directly with the electric and magnetic fields, one may therefore envisage that an immense number of photons in each phase space element are phased in such a manner as to create a quasi-stationary or large (macroscopic) wavelength structure, which behaves in a classical fashion according to Maxwell’s Equations. Conversely, at high frequencies, there are few photons in any phase space element and they behave in a statistically independent fashion consistent with the use of a single-particle distribution function $f_{\pm 1}$ or equivalently I_ν.

As we noted in §2, astrophysical systems are *not* in statistical equilibrium. While the material mean-free-paths between collisions shall always be smaller than any macroscopic dynamical scales of interest, the same shall not be true for photons, which can have inordinately large mean-free-paths. This distinction is precisely why we treat the material and the radiation differently. Deep in a stellar interior, for example, photons (like particles) will have extrememly short mean-free-paths and as we shall see in Scene 4, the use of angular and frequency moments of the specific intensity will essentially place the treatment of the radiation and the material on an equal footing.

5. The Script/Libretto

Operas are sung in many different languages, Italian, French, German and so forth, but the language in which RMHD is written is mathematics. If you cannot quantify something, you cannot really say anything about it. Algebra, topology and analysis are the essential cornerstones upon which the libretto is built.

Too often, the syntax and grammar of mathematics is offputting to the astrophysics student. It need not be this way. In these notes we have endeavored, sometimes to the point of tedium, to provide an acccessible and logical introduction to key mathematical concepts and definitions in plain english. We have also tried to employ a consistent and intuitive notation without undue complication. Perhaps we even succeed—you get to be the judge.

Finally, we have included an Appendix devoted entirely to the mathematics. There, we attempt to indicate how concepts and ideas build logically upon one another. We are not rigorous and rarely provide anything even remotely resembling a proof. These can be found elsewhere if necessary.

6. Further Reading

There are no monographs or review articles which cover all of the material in this *Opera*. Several come close and we mention just a few of them here. In providing sources for further reading, I am always guided by selecting those presentations that I have found especially useful or clear; often ones that I have used directly. There are, for example, very many monographs like Steven Shore's An Introduction to Astrophysical Hydrodynamics, which touch upon a lot of the topics presented in this *Opera*, but which does so in a superficial and not particularly satsifying fashion, from my perspective. I will generally omit these works on the grounds that life is often too short, and resources too limited, to spend on vehicles that go only part of the distance.

Materials that I have drawn from [read: plagiarized] in developing these notes are marked by a ⋆.

In large part, these notes rely most heavily upon:
⋆[**MM 1**] Dimitri Mihalas & Barbara Weibel Mihalas, Foundations of Radiation Hydrodynamics, (New York, NY: Oxford University Press; 1984), xv+718,
which is simply a superb treatment of radiating fluids. If you owned only one book, this would have to be it. There is now a Dover Publication edition that will not break the budget. This is a treasure trove of information and methods, with the only drawback being that it does not treat magnetohydrodynamics to any degree. For this you need to look elsewhere.

Elsewhere, is the two-volume set by Frank Shu
⋆[**S 1**] Frank H. Shu, The Physics of Astrophysics. Volume I: Radiation. Volume II: Gas Dynamics, (Mill Valley, CA: University Science Books; 1991),
which covers almost everything (and more) discussed here but in somewhat greater generality and with less emphasis on the formulation of RMHD per se. Never-the-less all the essential elements needed are here. The books are extremely insightful, the exposition elegant and concise. And if you could only own *three* books, you'd add these two to [**MM 1**] and would be pretty much set for the rest of your career in astrophysics. Here, I jest only slightly.

Perhaps one more book worth mentioning at this over-arching level is
⋆[**K 1**] Russell M. Kulsrud, Plasma Physics for Astrophysics, (Princeton, NJ: Princeton University Press; 2005), xviii+468,
which by comparison to the previous three tomes is very light on radiation but much more heavily invested in gravitation and magnetohydrodynamics from a wholistic fluid dynamics perspective. The presentation is exceptionally good for bringing out more of the underlying microphysics which I have swept far under the proverbial rug in this *Opera*.

For the treatment of energetic, i.e., non-thermal, particles by methods similar to transport theory used here for photons, an exhaustive, and at times exhausting, compendium is

[**S 2**] Reinhard Schlickeiser, Cosmic Ray Astrophysics, (Berlin, DE: Springer; 2002), vx+519,

while a very elegant and comprehensive formalism with numerous applications can be found in the two-volume set

[**M 1**] D.B. Melrose, Plasma Astrophysics: Nonthermal Processes in Diffuse Magnetized Plasmas. Volume I: The Emission, Absorption, and Transfer of Waves in Plasmas. Volume II. Astrophysical Applications, (New York, NY: Gordon and Breach Science Publishers; 1980). ix+269/viii+423.

Enjoy!