fairly well-defined large-scale component, for which something else than fast dynamo action must then be invoked. It turns out that the turbulent nature of the flow in the solar convective envelope can still do the trick, but to examine this we will need to adopt as statistical approach to turbulence and to the associated flow-field interactions. This is the focus of the following chapter.

Problems:

1. Calculate $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$ for the Roberts Cell flow. Confirm that it is a Beltrami flow, in the sense discussed in §3.1.

2. This problem aims at getting you to investigate in more detail what can happen to a magnetic field in the vicinity of a stagnation point. Consider the 2D cartesian incompressible flow defined by the stream function

$$\Psi(x, y) = u_0 xy \tag{3.30}$$

so that $u_x(x) = u_0 x$ and $u_y(y) = -u_0 y$ (note that the quantity u_0 has then units of s^{-1}!). We now want to consider the inductive action of this flow on a purely horizontal magnetic field, held fixed at values of $+B_0$ and $-B_0$, at $y = +L$ and $-L$ respectively. Evidently, this flow will tend to push the magnetic field towards the x-axis, where dissipation will occur since the field is oppositely directed on either side of the x-axis.

   (a) Show that the above flow has a stagnation point at the origin, and that its divergence is zero.

   (b) Show that in view of the imposed boundary condition, B_x can only be a function of the y coordinate everywhere in the domain $-L \leq y \leq +L$, $-\infty \leq x \leq \infty$.

   (c) Show that under these circumstances the x-component of the induction equation reduces to

$$\frac{1}{u_0}\frac{\partial B_x}{\partial t} = B_x + y\frac{\partial B_x}{\partial y} + \frac{\eta}{u_0 L}\frac{\partial^2 B_x}{\partial y^2},$$

   where lengths are expressed in units of L.

   (d) Show now that this equation accepts steady-state solutions of the form

$$B_x(y) = C\exp\left(-\alpha y^2\right) + D\exp\left(-\alpha y^2\right)\int_0^y \exp\left(\alpha (y')^2\right) \mathrm{d}y'$$

   where the parameter $\alpha = u_0 L^2/2\eta \equiv \mathrm{R}_m/2$ controls the relative importance of magnetic dissipation, as measured by the usual magnetic Reynolds number $\mathrm{R}_m = u_0 L/\eta$, and C and D are integration constants.

(e) Show that the assumed boundary conditions imply that $C = 0$ here;

(f) Now show that the thickness of the current sheet forming in the vicinity of $y = 0$ scales as $1/\sqrt{\alpha}$;

(g) Evaluate numerically the integral on the above solution for $B_x(y)$ and plot the variation of B_x as a function of y, for values of $\alpha = 10, 100$ and 10^3.

(h) Finally, compute the magnitude of the electric current in the z-direction, and show that the rate of energy dissipation is independent of the assumed value of η. Explain this physically.

3. The so-called ABC flow is another long-time candidate for fast dynamo action. It is a steady periodic flow in cartesian geometry, defined as

$$\mathbf{u}(x, y, z) = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$$

(a) Verify whether or not this is a Beltrami flow;

(b) Find the position(s) of the stagnation point(s) in the flow, for the specific case $A = B = C = 1$.

(c) Calculate a Poincaré section for this flow, using now parameter values $A = 1+$(your birth month/12), $B = 1+$(your birth day/30), $C = 1$. This involves repeatedly launching a particle somewhere on the $z = 0$ plane, and plotting its position at every crossing of $2\pi n$ planes in the z-direction ($n = 1, 2, ...$). Is this flow chaotic?

4. The flow near the 3D stagnation points in the ABC flow can be approximated in cylindrical polar coordinates (r, θ, z) by

$$\mathbf{u} = (\alpha r/2, 0, -\alpha z) \ ,$$

with $\alpha = \pm\sqrt{2}$.

(a) Calculate the the three Lyapunov exponents for $\alpha = +\sqrt{2}$ and $\alpha = -\sqrt{2}$, and show that in both cases their sum is zero.

(b) Obtain a solution to the steady ($\partial/\partial t = 0$) form of the induction equation, with $\mathbf{u}$ given by the above expression.

(c) On the basis of your solution, where would you expect to find magnetic fields in the flow?

(d) Again on the basis of your solution, estimate a length scale characterizing the thickness of the magnetic structures present in the solutions. How does this characteristic length scale with the magnetic Reynolds number?