Chapter 9

Mean-field theory

In my opinion nothing is contrary to nature
save the impossible, and that never happens.

Galileo Galilei
Discourses on Two New Sciences (1638; trans. S. Drake)

This chapter is concerned with the topic of mean-field electrodynamics, which encompasses the "classical" underpinning of dynamo theory from the period before the advent of of large super-computers and parallel-processing, when a megaflop was an over-budget Hollywood film that died on arrival at the box office. A number of themes which we have run across at earlier junctures in these notes reappear in this chapter in slightly different guises and with somewhat altered agendas. The principal achievement of these deliberations is some crucial physical insights—provided by the analytic mathematics upon which mean-field theory is based—on the operation of the α–effect, which is the cornerstone of nearly all astrophysical dynamos.[1]

9.1 Scale separation and statistical averages

The fundamental idea on which mean field theory rests is the **two scale approach**, which consists of a decomposition of the field variables into mean and fluctuating parts. This process naturally implies that an averaging procedure can meaningfully be defined. The derivation of mean field theory can proceed equally from the choice of space averages, time averages or ensemble averages. Space averages are somewhat easier to understand physically, and that is what we shall *implicitly* adopt here. Ensemble averages are more convenient from a purely mathematical perspective. It is the **ergodic hypothesis** which provides the physical and mathematical justification for our penchant of weaving back and forth between these various definitions of "$\langle\ \rangle$",

$$\langle A\rangle = \frac{1}{\lambda^3}\int_V A\, d\mathbf{x}, \qquad \text{or} \qquad \langle A\rangle = \frac{1}{\tau}\int A\, dt, \tag{9.1}$$

or the ensemble average.

We assume that the velocity and magnetic field can be decomposed into a mean and fluctuating part so that

$$\mathbf{U} = \langle\mathbf{U}\rangle + \mathbf{u}, \qquad \text{and} \qquad \mathbf{B} = \langle\mathbf{B}\rangle + \mathbf{b}. \tag{9.2}$$

[1]The material presented in the first three sections of this chapter was written by Thomas J. Bogdan, and is an abridged and modified variant of lecture notes prepared for the APAS-7500 course by Dr. Fausto Cattaneo (Department of Astronomy, University of Chicago) during the fall semester of 1994. The Cattaneo notes, in turn, were strongly inspired by the wonderful 1978 book *Magnetic field generation in electrically conducting fluids*, by H. K. Moffatt.

The decomposition (9.2) makes sense provided $\langle \mathbf{u} \rangle = \langle \mathbf{b} \rangle = 0$. The physical interpretation of (9.2) is as follows. The velocity and magnetic fields are characterized by a slowly varying component, $\langle \mathbf{U} \rangle$ and $\langle \mathbf{B} \rangle$, which vary on the characteristic large scale L, plus rapidly fluctuating parts, $\mathbf{u}$ and $\mathbf{b}$, which vary on the much smaller scale ℓ. The volume averages are computed over some intermediate scale λ such that

$$\ell \ll \lambda \ll L. \tag{9.3}$$

Whenever (9.3) is satisfied we say that we have a "good" scale separation.[2]

The objective of mean field theory is to produce a closed set of equations for the mean quantities. Substituting (9.2) into the induction equation (1.60), and averaging, we obtain equations for the mean and fluctuating quantities, namely

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \nabla \times \big(\langle \mathbf{U} \rangle \times \langle \mathbf{B} \rangle \big) + \nabla \times \boldsymbol{\mathcal{E}} + \eta \nabla^2 \langle \mathbf{B} \rangle, \tag{9.4}$$

and

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times \big(\langle \mathbf{U} \rangle \times \mathbf{b} \big) + \nabla \times \big(\mathbf{u} \times \langle \mathbf{B} \rangle \big) + \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b}, \tag{9.5}$$

where

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle, \qquad \text{and} \qquad \mathbf{G} = \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle. \tag{9.6}$$

The important thing is that (9.4) now contains a source term associated with the average of products of fluctuations. The term $\boldsymbol{\mathcal{E}}$, which is called the average electromotive force, or emf for short, plays a central role in this theory. Now, the whole point of the mean-field procedure is to avoid having to deal explicitly with the small scales, so we do not want to be integrating eq. (9.5) explicitly. But the we have a closure problem: eq. (9.4) is a 3-component vector equation, for the six components of $\langle \mathbf{B} \rangle$ and $\mathbf{b}$ (leaving the flow out of the picture for the moment). Therefore it is clear that to solve (9.4), $\boldsymbol{\mathcal{E}}$ must be expressed as some function of $\langle \mathbf{U} \rangle$ and $\langle \mathbf{B} \rangle$.

In order to obtain the the desired expression, we note that (9.5) is a *linear* equation for $\mathbf{b}$ with the term $\nabla \times \big(\mathbf{u} \times \langle \mathbf{B} \rangle \big)$ acting as a source. There must therefore exist a *linear* relationship between $\mathbf{B}$ and $\mathbf{b}$, and hence, one between $\mathbf{B}$ and $\langle \mathbf{u} \times \mathbf{b} \rangle$. The latter relationship can be expressed formally by the following series

$$\mathcal{E}_i = \alpha_{ij} \langle B \rangle_j + \beta_{ijk} \partial_k \langle B \rangle_j + \gamma_{ijkl} \partial_j \partial_k \langle B \rangle_l + \cdots, \tag{9.7}$$

where the tensorial coefficients, α, β, γ, and so forth must depend on $\langle \mathbf{U} \rangle$, what we might loosely term the *statistics* of the turbulent velocity fluctuations, $\mathbf{u}$, and on the diffusivity η— but *not* on $\langle \mathbf{B} \rangle$. In this sense, equations (9.4) and (9.7), constitute a closed set of equations for the evolution of $\langle \mathbf{B} \rangle$. The convergence of the series representation provided by equation (9.7) can be anticipated in those cases where the good separation of scales applies. For in these cases each successive derivative in equation (9.7) is smaller than the previous one by approximately a factor of $\ell / L \ll 1$. With any luck, we may expect equation (9.7) to be dominated by the first few terms.

9.2 The α–effect and turbulent diffusivities

We have already remarked that $\boldsymbol{\mathcal{E}}$ in (9.4) acts as a source term for the mean field. It is instructive to examine the contributions to $\boldsymbol{\mathcal{E}}$ deriving from the individual terms in the expansion (9.7). The first contribution is associated with the second-rank tensor, α_{ij}, thus

$$\mathcal{E}_i^{(1)} = \alpha_{ij} \langle B \rangle_j. \tag{9.8}$$

[2]In chapter 2 we used ℓ to denote the typical length over which $\mathbf{B}$ varies appreciably. Consistent with our scale separation hypothesis of this chapter, $\mathbf{B}$ is endowed with *two* characteristic length scales for the mean (L) and fluctuating (ℓ) constituents. The intermediate averaging length scale λ is related to the integration volume V by the obvious relation $\lambda \equiv V^{1/3}$.

The first thing to note is that α_{ij} must be a pseudo–tensor since it establishes a linear relationship between a polar vector–the mean emf, and an axial vector–the mean magnetic field. We can divide α_{ij} into its symmetric and antisymmetric parts, thus[3]

$$\alpha_{ij} = \alpha^s_{ij} - \epsilon_{ijk} a_k, \tag{9.10}$$

where $2a_k = -\epsilon_{ijk}\alpha_{ij}$. From (9.8) we have

$$\mathcal{E}^{(1)}_i = \alpha^s_{ij}\langle B\rangle_j + \big(\mathbf{a}\times\langle\mathbf{B}\rangle\big)_i. \tag{9.11}$$

The effect of the antisymmetric part is to provide an additional advective velocity (not in general solenoidal) so that the effective mean velocity becomes $\langle\mathbf{U}\rangle + \mathbf{a}$. The nature of the symmetric part is most easily illustrated in the case when $\mathbf{u}$ is an **isotropic** random field.[4] Then $\mathbf{a}$ is zero, α_{ij} must be an isotropic tensor of the form $\alpha_{ij} = \alpha\delta_{ij}$, and (9.11) reduces to

$$\boldsymbol{\mathcal{E}}^{(1)} = \alpha\langle\mathbf{B}\rangle. \tag{9.12}$$

Using Ohm's law, this component of the emf is found to generate a contribution to the mean current of the form

$$\mathbf{j}^{(1)} = \alpha\sigma_e\langle\mathbf{B}\rangle, \tag{9.13}$$

where σ_e is the electrical conductivity. For nonzero α, equation (9.13) implies the appearance of a mean current everywhere *parallel* to the mean magnetic field—the so-called α–effect. This is in sharp contrast to the more conventional case where the induced current $\sigma_e\big(\mathbf{U}\times\mathbf{B}\big)$ is *perpendicular* to the magnetic field. We are used to thinking as electrical currents being the source of magnetic fields (think of the Biot-Savart Law, of the pre-Maxwellian form of Ampère's Law); but a mechanically forced magnetic field can become a source of electrical current. That's really what induction is all about.

In the context of axisymmetric large-scale astrophysical magnetic fields, the importance of the α–effect is immediately apparent. We recall from our deliberations in §7.2.3 that a toroidal field could be generated from a poloidal one by differential rotation (velocity shear). The α–effect makes it possible to drive a mean toroidal current parallel to the mean toroidal field, which, in turn will regenerate a poloidal field thereby closing the dynamo cycle. This idea of inducing a toroidal current by the α–effect is at the heart of almost all models of astrophysical dynamos.

To appreciate the physical nature of the α–effect we pause to examine the original model of E.N. Parker (1955). We define a cyclonic event to be the rising of a fluid element associated with a definite circulation, say anticlockwise when seen from below (see Figure 9.1). In spherical geometry, we consider the effect of many such events on an initially purely toroidal field line (cf. Figure 9.2). Each cyclonic event creates an elemental loop of field with an associated current distribution that will have a component parallel to the initial field if the angle of rotation is less than π and antiparallel if it is greater. By assuming that the individual events are short lived we can rule out rotations of more than 2π. It is clear that the combined effect of many such events is to give rise to a net current with a component along $\langle\mathbf{B}\rangle$.

[3] Here, ϵ_{ijk} is the Levi-Civita tensor density, also known as the unit alternating tensor, and has the values $\epsilon_{ijk} = 0$ when i, j, k are not all different, $\epsilon_{ijk} = +1$ or -1 when i, j, k are all different and in cyclic, or acyclic, order respectively. A particularly useful formula is (Einstein summation over repeated indices in force):

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{9.9}$$

where δ_{ij} is the Kronecker-delta, and has the value $\delta_{ij} = 0$ if i, j are different, and $\delta_{ij} = 1$ when $i = j$.

[4] Throughout the rest of this chapter, we will have cause to repeatedly refer to the statistical properties of the turbulent velocity field. In order to avoid confusion we state the following definitions: a (random) field is **stationary** if its probability density function (pdf) is time independent, it is **homogeneous** if its pdf is independent of position, it is **isotropic** if its pdf is independent of orientation (or equivalently, invariant under rotations), and it is **reflectionally symmetric** if its pdf is invariant under parity reversal. We should note that isotropy and reflectional symmetry are taken here to be distinct properties, although this protocol is not universally accepted.

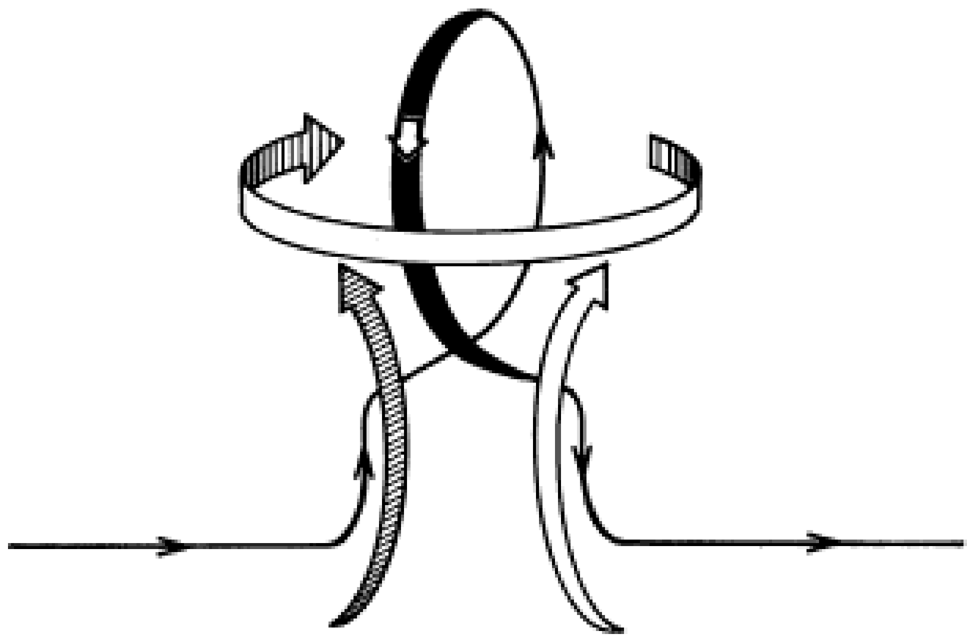

Figure 9.1: A sketch of magnetic line of force entrained by a cyclonic, rising fluid element in the frozen-in limit. Note that the resulting cyclonic loop can be viewed as resulting from an element of electric current flowing parallel to the original, uniform magnetic field. [*from:* Parker 1970, *The Astrophysical Journal*, vol. **162**, Figure 1].

An important property of α is its pseudoscalar nature, i.e. α changes sign under parity transformations. This implies that α can be nonzero only if the statistics of $\mathbf{u}$ lacks reflectional symmetry. In other words the velocity field must have a definite handedness (also called chirality). In the example above there is a definite relationship between vertical displacements and sense of circulation.[5] In general the lack of reflectional symmetry of the fluid velocity manifests itself through a nonzero value of the fluid *helicity*, $\langle \mathbf{u}\cdot(\nabla\times\mathbf{u})\rangle$, itself a pseudo scalar. As we shall presently see there is an important relation between fluid helicity and the α–effect.

We now turn to the next term in the expansion (9.7), namely

$$\mathcal{E}_i^{(2)} = \beta_{ijk}\partial_k\langle B\rangle_j. \tag{9.14}$$

The physical interpretation of the third-rank pseudotensor, β_{ijk}, is again most easily gained when $\mathbf{u}$ is isotropic, and so we dispense with general considerations and cut straight to the chase. For isotropic turbulence, it follows that, $\beta_{ijk} = \beta\epsilon_{ijk}$, where β is a scalar, and so we have

$$\nabla\times\mathcal{E}^{(2)} = \nabla\times\big(-\beta\nabla\times\langle\mathbf{B}\rangle\big) = \beta\nabla^2\langle\mathbf{B}\rangle. \tag{9.15}$$

We recognize the scalar β as an additional contribution to the effective diffusivity of $\langle\mathbf{B}\rangle$, which thus becomes $\eta_e \equiv \eta+\beta$. In cases where $\beta \gg \eta$ one refers to $\eta_e \approx \beta$ as the **turbulent diffusivity**.

In summary, our heuristic treatment of mean-field electrodynamics has led us to an evolution equation for the large-scale magnetic field, $\langle\mathbf{B}\rangle$, which takes account of coherences between fluctuation-fluctuation interactions of the small-scale turbulent magnetic and velocity fields. For homogeneous, stationary, and isotropic velocity turbulence, this equation assume the particularly elegant and physically intuitive form

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \nabla\times\big(\langle\mathbf{U}\rangle\times\langle\mathbf{B}\rangle\big) + \alpha\nabla\times\langle\mathbf{B}\rangle + (\eta+\beta)\nabla^2\langle\mathbf{B}\rangle. \tag{9.16}$$

[5] Why?

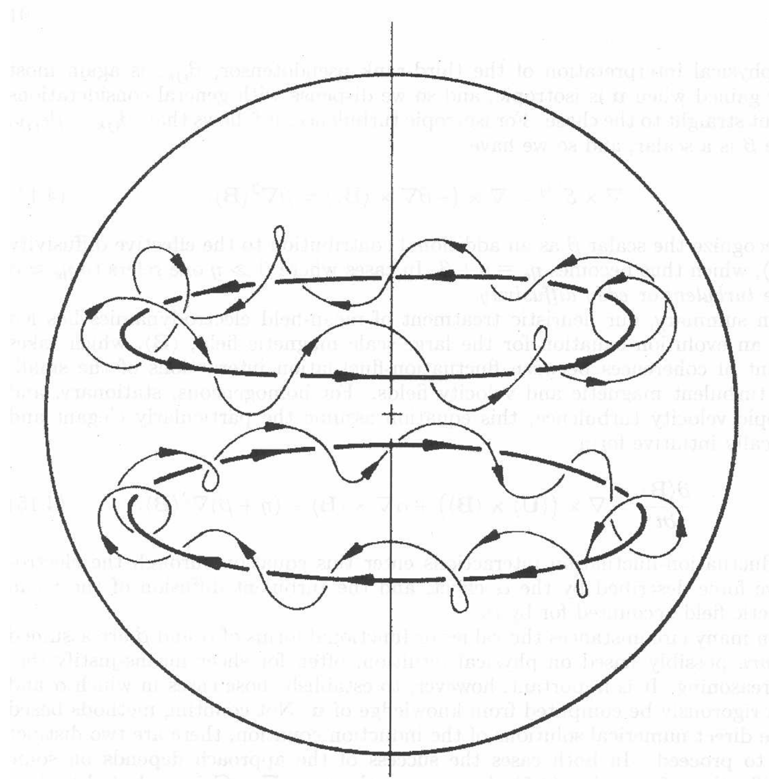

Figure 9.2: A sketch of the azimuthal (toroidal) magnetic lines of force (heavy lines) in the northern and southern hemisphere, carried into spirals by local cyclonic convection cells (thin lines). The collective effect of these events is a mean electric current flowing in the azimuthal direction, which can sustain a poloidal magnetic component. [*from:* Parker 1979, *Cosmical Magnetic Fields*, (Oxford: Clarendon Press), p. 548.]

The fluctuation-fluctuation interactions enter this equation through the electromotive force described by the α–effect, and the turbulent diffusion of the mean magnetic field accounted for by β.

In many circumstances the values or functional forms of α and β are assumed *a priori*, possibly based on physical intuition, often for sheer means-justify-the-ends reasoning. It is important, however, to establish those cases in which α and β can rigorously be computed from knowledge of $\mathbf{u}$. Not counting methods based on the direct numerical solutions of the induction equation, there are two distinct ways to proceed. In both cases the success of the approach depends on some simplification of equation (9.5). In one case the term $\nabla \times \mathbf{G}$ is neglected leading to the so-called first order smoothing approximation (FOS). In the other, the term $\eta\nabla^2\mathbf{b}$ is neglected, leading to the Lagrangian approximation. The two approaches are complementary in the sense that the former is applicable (for most physically relevant circumstances) when the diffusivity is large and the latter when it is small.

9.2.1 First order smoothing

We begin with the case where $\nabla\times\mathbf{G}$ may be neglected. Assuming that $\langle\mathbf{U}\rangle = 0$, (9.5) becomes[6]

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times \big(\mathbf{u} \times \langle\mathbf{B}\rangle\big) + \nabla \times \mathbf{G} + \eta\nabla^2\mathbf{b}, \tag{9.17}$$

[6]This is permissible since we can always effect a Galilean transformation into the comoving frame of the mean flow.

$$\mathcal{O}(b_o/\tau) \quad \mathcal{O}(B_o u_o/\ell) \quad \mathcal{O}(u_o b_o/\ell) \quad \mathcal{O}(\eta b_o/\ell^2) \tag{9.18}$$

where the magnitudes of the terms in (9.17) are as indicated. Here ℓ and τ are the characteristic length and time scales associated with $\mathbf{u}$, and u_o, b_o and B_o are the rms values of $\mathbf{u}$, $\mathbf{b}$, and $\langle\mathbf{B}\rangle$. Two distinct situations are of physical interest:

$$\tau \approx \ell/u_o \ , \tag{9.19}$$

$$\tau \ll \ell/u_o \ . \tag{9.20}$$

The first case corresponds to conventional fluid turbulence where the characteristic time, or the correlation time, is comparable with the eddy-turnover time. In the second case, the correlation time is much less than the turnover time. This corresponds, for example, to an ensemble of random waves. This latter case is sometimes also referred to as the Markovian approximation.

If (9.20) is satisfied, then $|\nabla\times\mathbf{G}| \ll |\partial_t \mathbf{b}|$, and then to a good approximation,

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla\times\big(\mathbf{u}\times\langle\mathbf{B}\rangle\big) + \eta\nabla^2\mathbf{b}, \tag{9.21}$$

is valid. If, on the other hand, it is equation (9.19) that is satisfied, then $|\nabla\times\mathbf{G}|$ and $|\partial_t\mathbf{b}|$ are necessarily of the same order. Our basic goal is to find a way to discard the $\nabla\times\mathbf{G}$ term since it leads to a very complicated equation for $\mathbf{b}$. We notice that both $|\nabla\times\mathbf{G}|$ and $|\partial_t\mathbf{b}|$ are negligible compared to $\eta\nabla^2\mathbf{b}$ if we can assume that

$$r_m = \frac{u_o\ell}{\eta} \ll 1, \tag{9.22}$$

where r_m is the magnetic Reynolds number that pertains to the *small-scale* magnetic fluctuations. While we have repeatedly stressed that the magnetic Reynolds number for the large-scale magnetic field is necessarily a very large number in most astrophysical applications, owing to the large values for L, it is not quite so obvious that r_m should also be much in excess of unity. If we accept for the moment that ℓ *may* be sufficiently small that equation (9.22) is valid, then equation (9.17) reduces to

$$0 = \nabla\times\big(\mathbf{u}\times\langle\mathbf{B}\rangle\big) + \eta\nabla^2\mathbf{b} \ . \tag{9.23}$$

For all intents and purposes, both of these limiting arguments lead to equation (9.21), since equation (9.23) is basically contained within equation (9.21) as a further special case. In either example, therefore, fluctuations in $\mathbf{b}$ are generated solely by the interaction of the random velocity $\mathbf{u}$ with the mean field $\langle\mathbf{B}\rangle$, and fluctuation–fluctuation interactions, described by the $\nabla\times\mathbf{G}$ term can safely be neglected. A little thought reveals that the success of the present approach hinges on the existence of a short memory time. In case (9.20) the correlation time of the turbulence is short, so that the effects of past history are small. In case (9.19) the further requirement that $r_m \ll 1$ ensures that diffusion acts quickly enough to remove any effects of past history, even though the turbulence per se now has a rather long memory. As we shall see serious difficulties can arise when the memory time is not small.

With these remarks being said, our next task is to solve equation (9.21) within the volume $V = \lambda^3$, for a specified (turbulent) velocity field, $\mathbf{u}$, and a prescribed (effectively) constant mean magnetic field, $\langle\mathbf{B}\rangle$. Of course, it is not $\mathbf{b}$ per se that is of interest, but rather the mean emf $\boldsymbol{\mathcal{E}}$ generated within the volume V. Hence it will prove necessary to specify the statistical properties of $\mathbf{u}$, so that α and β can be related to them. Within FOS only second-order moments of $\mathbf{u}$ are required, which can be specified entirely in terms of a beast called the velocity spectrum tensor. The mathematics gets rather intricate, and those having never seen an octuple integral are encourage to consult §X.Y of the monograph my Moffatt listed in the bibliography at the end of this chapter.

Rather than work out general expressions for the α and β tensors, to better appreciate some of the problems to be encountered within FOS in the limit of small η we examine a particularly simple example. Consider the following velocity field consisting of a single helical wave:

$$\mathbf{u}(\mathbf{x},t) = u_o\big(\sin(kz-\omega t), \cos(kz-\omega t), 0\big) = \text{Re}\left\{\mathbf{u}_o \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right\}, \tag{9.24}$$

where

$$\mathbf{u}_o = u_o(-\mathrm{i}, 1, 0), \qquad \mathbf{k} = (0,0,k). \tag{9.25}$$

For this velocity field

$$\nabla\times\mathbf{u} = k\mathbf{u}, \quad \mathbf{u}\cdot(\nabla\times\mathbf{u}) = ku_o^2, \quad \text{and} \quad \mathrm{i}\mathbf{u}_o\times\mathbf{u}_o^* = 2u_o^2(0,0,1)\ . \tag{9.26}$$

The corresponding periodic solution of (9.21) has the form

$$\mathbf{b}(\mathbf{x},t) = \text{Re}\left\{\mathbf{b}_o \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right\}, \quad \text{with} \ \ \mathbf{b}_o = \frac{\mathrm{i}\langle\mathbf{B}\rangle\cdot\mathbf{k}}{-\mathrm{i}\omega+\eta k^2}\mathbf{u}_o. \tag{9.27}$$

Hence we can obtain

$$\boldsymbol{\mathcal{E}} = \langle\mathbf{u}\times\mathbf{b}\rangle = -\frac{\eta u_o^2(\langle\mathbf{B}\rangle\cdot\mathbf{k})k^2}{\omega^2+\eta^2k^4}(0,0,1)\ , \tag{9.28}$$

which gives

$$\alpha_{ij} = \alpha^{(3)}\delta_{i3}\delta_{j3}, \quad \alpha^{(3)} = -\frac{\eta u_o^2 k^3}{\omega^2+\eta^2k^4}\ . \tag{9.29}$$

In the example above, we should note that $\mathbf{u}\times\mathbf{b}$ is uniform, therefore $\mathbf{G}$ is zero, and the FOS approximation is *exact*. Expression (9.29) then states that $\alpha\to 0$ as $\eta\to 0$, and that some diffusion is *necessary* for the α–effect to work. In order to appreciate some additional subtle effects associated with η, we note that the above solution does not satisfy the nominal initial condition, $\mathbf{b}(\mathbf{x},0)=0$. If we insist that this condition be satisfied, then we must add to the particular solution (9.27), a transient term of the form

$$\mathbf{b}_1 = -\text{Re}\left\{\mathbf{b}_0 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{e}^{-\eta k^2 t}\right\}, \tag{9.30}$$

which is simply a magnetic diffusion mode of the homogeneous equation. This additional term also contributes to $\boldsymbol{\mathcal{E}}$, and therefore to α. This transient contribution will decay to zero in a time $\mathcal{O}(\eta k^2)^{-1}$, and, clearly, the memory of the initial conditions will then be forgotten after a time $t\geq(\eta k^2)^{-1}$. However, the limit $\eta\to 0$ poses some interesting problems. If we fix η to some small positive value and let $t\to\infty$ then the transient disappears and we recover (9.29). If, on the other hand, we *first* let $\eta\to 0$, and *then* try to ascertain the long-time behavior, we have

$$\boldsymbol{\mathcal{E}} = \langle\mathbf{u}\times\mathbf{b}\rangle = -\frac{1}{\omega}u_o^2 k\sin\omega t\ (0,0,1)\ . \tag{9.31}$$

The mean emf $\boldsymbol{\mathcal{E}}$, and therefore α, never settles down to any definite value as $t\to\infty$. For this latter case the initial conditions are never forgotten.

9.2.2 The Lagrangian approximation

We saw that in the limit of small diffusivity the FOS approximation cannot consistently be used for standard turbulence and for the case of random waves it may run into difficulties if zero frequency waves are present. It is therefore desirable to derive another approximation that does not require the neglect of the $\nabla\times\mathbf{G}$ term in equation (9.17). This is the basis of the Lagrangian approximation which retains the $\nabla\times\mathbf{G}$ term but neglects instead the diffusive term $\eta\nabla^2\mathbf{b}$. Clearly the Lagrangian approximation may most likely be justified in the limit of vanishing η.

The Lagrangian approximation leads to expressions for α and β in terms of second order statistics of the *Lagrangian* velocity field. Since these are less commonly used in turbulence work than their Eulerian counterparts, it is instructive to begin with a simpler case and examine the diffusion of a passive scalar, as was first considered by G.I. Taylor (1921). Let θ be a scalar quantity advected by the random velocity field $\mathbf{u}$. Then the evolution of θ is governed by

$$\frac{\partial\theta}{\partial t}+\mathbf{u}\cdot\nabla\theta=0\ , \tag{9.32}$$

Again we assume that the velocity correlation length is ℓ and define averages over some scale $\lambda\gg\ell$. We anticipate that the evolution of $\langle\theta\rangle$ will be governed by a diffusion equation of the type

$$\frac{\partial\langle\theta\rangle}{\partial t}=\kappa_e\nabla^2\langle\theta\rangle\ , \tag{9.33}$$

where κ_e is the effective, or turbulent diffusivity. For homogeneous, isotropic turbulence, we expect further that κ_e will be a scalar satisfying $\kappa_e=\mathcal{O}(u_o\ell)$. The physical basis for this expectation is that the effects of turbulent motions are to convect the quantity θ over a distance ℓ at a typical velocity u_o. We notice a similarity between this argument and the procedure used in kinetic theory of gases to compute the *collisional* diffusivity in terms of the mean free path and velocity distribution function.

The solution of equation (9.32) can be developed through a great many nefarious means. Of all of these possibilities, by far the most efficient is to recognize that equation (9.32) is identical to the continuity equation for a solenoidal flow field, e.g., equation (I.1.3). In our discussion of the Lagrangian formulation of wave propagation, in §III.2.3, we devised a means to integrate the continuity equation that was even valid under the more general circumstance in which $\nabla\cdot\mathbf{u}\neq 0$. This method hinged upon viewing the dynamics as a mapping

$$\mathbf{x}(\mathbf{a},t)=\mathbf{a}+\boldsymbol{\xi}(\mathbf{a},t)\ , \tag{9.34}$$

which takes an element of fluid situated at the point $\mathbf{a}$ at time $t=0$, to the point $\mathbf{x}$ at any subsequent time $t\geq 0$.[7]Two points are worth mentioning. First, we use $\mathbf{a}$ here instead of $\mathbf{x}^\star$ to represent the initial location. Notation, notation, notation! Second, I have finally outsmarted TEX and figured out how to boldface greek letters! And then equations (III.2.54)–(III.2.58) give the so-called *Cauchy solution* to the problem in terms of the Jacobian of the mapping

$$\mathcal{J}_{ij}(\mathbf{a},t)=\frac{\partial x_i}{\partial a_j}=\delta_{ij}+\frac{\partial\xi_i}{\partial a_j}\ . \tag{9.35}$$

Since we have specialized our discussion to strictly solenoidal flows, it follow that $\mathcal{J}\equiv\det\big(\mathcal{J}_{ij}\big)=1$, and so from equation (III.2.54) we find,

$$\theta\big(\mathbf{x}(\mathbf{a},t),t\big)=\theta(\mathbf{a},0)\ , \tag{9.36}$$

where $\mathbf{a}$ is the initial position of the fluid trajectory that passes through $\mathbf{x}$ at time t. Recall from our extensive discussion presented in §III.2.3 that equation (4.54) is a Lagrangian statement. The analogous Eulerian statement requires and inversion of the dynamic mapping. For short times, or equivalently, small Lagrangian displacements $|\boldsymbol{\xi}|=|\mathbf{x}-\mathbf{a}|\leq\ell$, we can Taylor-expand equation (9.36) to obtain the (approximate) corresponding Eulerian statement,

$$\theta(\mathbf{x},t)=\theta(\mathbf{x},0)-\xi_i\partial_i\theta(\mathbf{x},0)+\frac{1}{2}\xi_i\xi_j\partial_i\partial_j\theta(\mathbf{x},0)+\cdots \tag{9.37}$$

[7] 14

Ensemble-averaging (9.37) and assuming that there are no initial correlations between θ and $\mathbf{u}$ we obtain

$$\langle\theta(t)\rangle = \langle\theta(0)\rangle + \frac{1}{2}\langle\xi_i\xi_j\rangle\partial_i\partial_j\langle\theta(0)\rangle + \cdots . \tag{9.38}$$

For isotropic flow we may further simplify (9.38) to get

$$\langle\theta(t)\rangle = \langle\theta(0)\rangle + \frac{1}{6}\langle\xi^2\rangle\nabla^2\langle\theta(0)\rangle + \cdots . \tag{9.39}$$

After a correlation time, deviations from the initial configuration will become substantial and the square displacement field will behave like a random walk, i.e

$$\langle\xi^2\rangle \sim t. \tag{9.40}$$

In this regime, (9.39) can be regarded as a solution of the diffusion equation (9.33) (in perturbation theory) with

$$\kappa_e = \frac{1}{6}\frac{d}{dt}\langle\xi^2\rangle \ . \tag{9.41}$$

It is also useful to express the diffusivity in terms of velocity correlations. This can easily be achieved by noting that

$$\xi_i = \int_0^t v_i(\mathbf{a}, t')dt', \tag{9.42}$$

where $v_i(\mathbf{a}, t)$ is the Lagrangian velocity. Then

$$\langle\xi^2\rangle = \int_0^t\int_0^t \langle\mathbf{v}(\mathbf{a}, t_1)\cdot\mathbf{v}(\mathbf{a}, t_2)\rangle dt_1 dt_2 = 2\int_0^t \big[tR_L(s) - sR_L(s)\big]ds$$

$$\approx t\int_{-\infty}^{+\infty} R_L(s)ds \ , \tag{9.43}$$

$$R_L(s) \equiv \langle\mathbf{v}(\mathbf{a}, t)\cdot\mathbf{v}(\mathbf{a}, t+s)\rangle. \tag{9.44}$$

In order to derive (9.43) we have assumed that for stationary turbulence the correlation function depends on the time difference $|t_1 - t_2|$ but not on t_1 or t_2 separately. Furthermore we also assumed that most of the contributions to the last integral come from $s \sim 0$. Both assumptions are believed to be justified for turbulent flows. The last integral in (9.43) is equal to the zero frequency component of the Lagrangian energy spectrum, and so we obtain another useful expression for the diffusivity, namely

$$\kappa_e = \frac{1}{6}\Phi_L(0) \ . \tag{9.45}$$

Where the energy spectrum is defined as:

$$\Phi_L(\omega) = \int_{-\infty}^{\infty} R_L(t)e^{-\mathrm{i}\omega t} \ \ dt \ . \tag{9.46}$$

Having practiced on the scalar case we are ready to tackle the more complicated case of the magnetic field. The Cauchy solution for the magnetic field reads [cf. equation (III.2.56)]

$$B_i(\mathbf{x}, t) = \frac{\partial x_i}{\partial a_j}B_j(\mathbf{a}, 0) \ , \tag{9.47}$$

which is the vector equivalent of (9.36). It shows that the magnetic field is both advected and stretched by the velocity field. From equation (9.47) we can immediately calculate the emf, namely

$$\mathcal{E}_i = \langle \mathbf{u} \times \mathbf{b} \rangle_i = \langle \mathbf{u} \times \mathbf{B} \rangle_i = \epsilon_{ijk} \Big\langle v_j(\mathbf{a},t) B_l(\mathbf{a},0) \frac{\partial x_k}{\partial a_l} \Big\rangle \ . \tag{9.48}$$

The calculation of α follows from (9.48) most simply if we assume that $\langle \mathbf{B} \rangle$ is uniform (and therefore constant), and that $\mathbf{b}(\mathbf{x},0) = 0$, so that $\mathbf{B}(\mathbf{a},0) = \langle \mathbf{B} \rangle$. Then

$$\alpha_{il}(t) = \epsilon_{ijk} \Big\langle v_j(\mathbf{a},t) \frac{\partial x_k(\mathbf{a},t)}{\partial a_l} \Big\rangle, \tag{9.49}$$

where now α_{il} is explicitly a function of time. As before, we use (9.42) to express (9.49) in terms of velocities. We get

$$\alpha_{il}(t) = \epsilon_{ijk} \int_0^t \Big\langle v_j(\mathbf{a},t) \frac{\partial v_k(\mathbf{a},s)}{\partial a_l} \Big\rangle ds. \tag{9.50}$$

The time dependence derives from the requirement that $\mathbf{b}(\mathbf{x},0) = 0$ which trivially implies that $\alpha(0) = 0$. For times longer than the correlation time we again expect that the imprint of the initial conditions should be forgotten and that α should rapidly approach its asymptotic value. In other words we expect that as in (9.43) we may carry the integration to infinity and write

$$\alpha_{il} \approx \epsilon_{ijk} \int_0^\infty \Big\langle v_j(\mathbf{a},t) \frac{\partial v_k(\mathbf{a},s)}{\partial a_l} \Big\rangle ds. \tag{9.51}$$

There are however some important differences between the integrand of (9.43) and that of (9.51) that may severely undermine the convergence of the integral in (9.51). The problem is associated with the long time behavior of the derivative in the correlation term in (9.51), namely

$$\frac{\partial v_k}{\partial a_l} = \Big(\frac{\partial v_k}{\partial x_m} \Big) \Big(\frac{\partial x_m}{\partial a_l} \Big) \ . \tag{9.52}$$

The first term on the LHS of (9.52) is in general stationary for stationary turbulence, however the second is not, since two initially adjacent particles tend to drift apart so that $|\delta \mathbf{x}|/|\delta \mathbf{a}| \sim t^{1/2}$ as $t \to \infty$. It follows that the integrand of (9.51) is both a function of t and s and not of s alone as in (9.43). Expression (9.52) was obtained for zero diffusivity, the convergence of (9.51) in the limit of $\eta \to 0$ is still largely an open question.

For isotropic turbulence (9.51) simplifies to

$$\alpha(t) = -\frac{1}{3} \int_0^\infty \big\langle \mathbf{v}(\mathbf{a},t) \cdot \big(\nabla^{(\mathbf{a})} \times \mathbf{v}(\mathbf{a},s) \big) \big\rangle ds \ , \tag{9.53}$$

where the differentiation is with respect to $\mathbf{a}$. We may interpret the integrand as a Lagrangian helicity correlation.

Assuming that the initial mean field has a uniform *gradient* and that $\mathbf{b}(\mathbf{x},0) = 0$ leads to a rather similar calculation for the diffusivity β. In the isotropic case we have

$$\beta(t) = \frac{1}{3} \int_0^t \langle \mathbf{v}(t) \cdot \mathbf{v}(s) \rangle ds + \int_0^t \alpha(t) \alpha(s) ds$$

$$+ \frac{1}{6} \int_0^t \int_0^t \big\langle \mathbf{v}(t) \cdot \mathbf{v}(s_2) \nabla^{(\mathbf{a})} \cdot \mathbf{v}(s_1) - \big(\mathbf{v}(t) \cdot \nabla^{(\mathbf{a})} \mathbf{v}(s_1) \big) \cdot \mathbf{v}(s_2) \big\rangle ds_1 ds_2. \tag{9.54}$$

The first term in (9.54) is identical to the expression for a passive scalar, the second and third terms are associated with the vector character of the field $\mathbf{B}$. In particular the term involving products of α at different times suggests that helicity fluctuations may play an important role.[8] The convergence of the term involving triple Lagrangian correlations is open to the same doubts as (9.51). It is important to note that (9.54) implies that β may have a *negative* value. That being the case, and further if $\eta + \beta < 0$ then the effects of the diffusion term are to *amplify* rather than suppress high frequency components. This behavior is probably incompatible with the two scale approach used to derive (9.54).

9.2.3 Higher-order approximations and numerical simulations

Obviously, calculating the alpha-effect and turbulent diffusivity is not a simple affair. Even the two most severe simplifying assumptions we considered above did note exactly lead to simple mathematics, and to add insult to injury the parameter regimes for which these simple development are expected to hold do not square well with what we think we know about solar interior conditions. The closest we can get to the Sun, in a tractable manner, is the so-called Second-Order Correlation Approximation (SOCA), which neglects cross-correlations between the different velocity components but retains the possibility that the intensity of turbulence itself can vary with position. Under this assumption of near-isotropy, we then have

$$\langle u_j u_k \rangle = \frac{1}{3}\langle \mathbf{u}^2 \rangle \delta_{jk} \ . \tag{9.55}$$

This leads to simple diagonal forms for the α and β tensors:

$$\alpha = -\frac{1}{3}\,\tau_c \langle \mathbf{u}\cdot(\nabla\times\mathbf{u})\rangle \ , \tag{9.56}$$

$$\beta = \frac{1}{3}\,\tau_c \langle \mathbf{u}^2 \rangle \ , \tag{9.57}$$

where τ_c is the correlation time for the turbulent flow. Equation (9.56) tells us that the α-effect is a direct function of the helicity of turbulent component of the flow; think back of Parker's picture of twisted magnetic fieldlines (Fig. 9.1) and convince yourself that this is indeed how it whould be for the "cartoon" to work. Equation (9.57) takes on the same form that can be rigorously derived in the case rigorously homogeneous and isotropic turbulence; it tells us that the turbulent diffusivity is more efficient when the turbulent is more vigorous, which also makes intuitive sense since, in order to destroy the magnetic field by folding, the flow must do work work against the Lorentz force.

An entirely different line of attack is to carry out MHD numerical simulations of turbulent flows including an externally-imposed magnetic field, and from the simulation statistics compute α and β directly from the simulations. There has been many such simulations, which, almost surprisingly, have corroborated the expressions obtained from SOCA. In particular, in weakly non-homogeneous turbulence the simulation results are compatible with an expression of the form:

$$\alpha = -\frac{1}{3}\,\tau_c \left(\langle \mathbf{u}\cdot(\nabla\times\mathbf{u})\rangle - \frac{1}{\rho}\langle \mathbf{j}\cdot\mathbf{B}\rangle \right) \ . \tag{9.58}$$

Notice that the second term on the RHS, corresponding to the current helicity associated with the small-scale magnetic field, has a sign opposite to that kinetic helicity. This says, in essence, that the Lorentz force opposes the twisting of the large-scale magnetic field by the turbulent flow, which makes good physical sense.

9.2.4 Heuristics

The fact remains that more often that not, and certainly in all mean-field dynamo models to be considered in the next chapter, the mean-field coefficients α and β will be chosen *a priori*,

[8]For a discussion of the last point see Kraichnan (1976)

although we will take care to embody in these choices what we have learned from our beirf excursion into mean-field theory. In fact we have done so already in posing the mathematicval form for the net magnetic diffusivity used in §§7.1 and 7.2.3, when we used a much larger magnetic diffusivity in the convective envelope than in the underlying radiative core, reflecting the existence of strong turbulence in the former but not in the latter. We will be doing the same with the turbulent α-effect, by concentrating it in the lower part of the convection zone; this is because the turbulence is expected to be most helical because its turnover time is commensurate with the solar rotation period, so that the Coriolis force can most efficiently break the reflection symmetry of turbulence, required to get a non-zero α-effect. We will also insist that the α-effect changes sign across solar hemisphere, to reflect the hemispheric dependence of the Coriolis force; and other things like that.

What this will mean is that our dynamo models will now have a descriptive, rather than predictive value. We will be picking turbulent dynamo coefficient that "do the right thing" for the Sun, and see how the resulting models behave as we change other aspects of the model, or apply them to stars other than the Sun. As the following example will show, we can still learn a lot from mean-field electrodynamics, even though we have foregone physical and mathematical determinism.

9.3 Dynamo waves

Having derived the mean field dynamo equations and having established that, at least in some regimes, the α and β coefficients are well behaved, it is instructive to study some elementary solutions. We distinguish different types of solution in terms of the dominant regenerative processes. Although the distinction applies in general, it is most easily illustrated in a simplified Cartesian geometry.

To this end, we begin by recalling our quintessential mean-field equation in the presence of homogeneous, stationary and isotropic turbulence,

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \nabla\times\big(\langle\mathbf{U}\rangle\times\langle\mathbf{B}\rangle\big) + \alpha\nabla\times\langle\mathbf{B}\rangle + (\eta+\beta)\nabla^2\langle\mathbf{B}\rangle. \tag{9.59}$$

The simplest Cartesian problem which comes equipped with all the standard features of the fancy mean-field astrophysical dynamos we shall presently contemplate in chapter 10, arises from the basic shear flow

$$\langle\mathbf{U}\rangle = \Omega z\ \hat{\mathbf{e}}_y\ , \tag{9.60}$$

where Ω is a constant [units: s^{-1}]. We shall further assume that the mean-field coefficients α [units: m s^{-1}] and $\eta_e = \beta+\eta$ [units: $\mathrm{m}^2\ \mathrm{s}^{-1}$] are constant.

We begin by uncurling equation (9.16) to obtain[9]

$$\left\{\frac{\partial}{\partial t} + \Omega z\frac{\partial}{\partial y} - \eta_e\nabla^2\right\}\langle\mathbf{A}\rangle = \alpha\langle\mathbf{B}\rangle - \Omega\big(\hat{\mathbf{e}}_y\cdot\langle\mathbf{A}\rangle\big)\hat{\mathbf{e}}_z. \tag{9.61}$$

The two terms on the RHS of this equation parameterize the α–effect and the Ω–effect. Recall that the Ω–effect describes generation of a toroidal magnetic field by the shearing out of a poloidal field. The (mean-field) α–effect accounts for the regeneration of *both* poloidal and toroidal magnetic fields due to the chirality, or handedness, of the turbulent flow field. These two terms offer the possibility of dynamo action overcoming the magnetic diffusion term which resides on the LHS of this equation. We shall soon see that dynamo action is possible in the absence of shear ($\Omega = 0$), leading to what is called an α^2–dynamo. When both α and Ω are

[9]Notice that in uncurling equation (9.16) we have made astute use of our ability to add the net divergence of any scalar to the RHS of equation (9.60). What scalar did we pick? Is this at all related to our freedom to to make a gauge transformation in choosing a representation for the vector potential?

nonzero we have an $\alpha\Omega$–dynamo. And when only Ω is nonzero we have—well, no dynamo at all![10]

Equation (9.61) is very nearly another example of a PDE with constant coefficients. The offending term is the advective derivative $\Omega z\partial/\partial y$. One means to circumvent the phase-mixing and related chicanery this term has waiting in the wings for us (cf. Figure 7.9) is to focus our attention of two-dimensional dynamo waves which are invariant under translation in the streamwise direction (i.e., $\partial/\partial y \equiv 0$). With the advective term summarily dealt with, we are now free to look for elementary plane-wave solutions of the form

$$\langle \mathbf{A} \rangle = \mathbf{a}_0 \exp\bigl[\lambda t + \mathrm{i}k(z\cos\vartheta + x\sin\vartheta)\bigr] \ . \tag{9.62}$$

We may assume that $k \geq 0$ and $0 \leq \vartheta \leq 2\pi$ are prescribed (real) parameters. If equation (9.62) is substituted into equation (9.61), the requirement that there be nontrivial $\mathbf{a}_0$ eigenvectors leads to the dispersion relation,[11]

$$\bigl(\lambda + \eta_e k^2\bigr)^2 = \alpha k\bigl(\alpha k + \mathrm{i}\Omega\sin\vartheta\bigr) \ . \tag{9.63}$$

Equation (9.63) provides us with a quadratic equation for λ, with the two solutions,[12]

$$\lambda_\pm = \ -\eta_e k^2 \pm \sqrt{\frac{|\alpha|k}{2}} \quad \Biggl\{ \Bigl(\sqrt{\Omega^2\sin^2\vartheta + \alpha^2 k^2} + |\alpha|k\Bigr)^{\frac{1}{2}}$$

$$+\mathrm{i}\ \mathrm{sign}(\Omega\alpha\sin\vartheta)\ \Bigl(\sqrt{\Omega^2\sin^2\vartheta + \alpha^2 k^2} - |\alpha|k\Bigr)^{\frac{1}{2}} \Biggr\} \ . \tag{9.64}$$

The λ_- solution can only produce a disturbance which decays with the passage of time, and so the possibility of an exponentially growing mean-field rests on the properties of the λ_+ root. Dynamo action occurs when $\mathrm{Re}(\lambda_+) > 0$. Examination of equation (9.64) indicates that an exponentially growing *dynamo wave* obtains when $0 < k < k_\star$, where the critical wavenumber $k_\star$ is one of the (six) roots of the equation,

$$k_\star^6 - \frac{\alpha^2}{\eta_e^2}k_\star^4 - \frac{\alpha^2\Omega^2}{4\eta_e^4}\sin^2\vartheta = 0 \ , \tag{9.65}$$

If $k_\star \to 0$ then the "window" for dynamo action disappears. This occurs when $\alpha \to 0$, which confirms that there is no such beast as an Ω^2–dynamo. From a physical perspective it makes a good deal of sense that the dynamo window inhabits the small-wavenumber, large-wavelength, end of the range of possible parameters. Clearly dynamo waves with rapid spatial fluctuations are susceptible to severe damping due to the enhanced diffusivity $\eta_e \approx \beta$. On the other hand, if the spatial variations of $\langle \mathbf{A} \rangle$ are too large, then there is very little $\langle \mathbf{B} \rangle$ for the α–effect to work on, and so the dynamo process again stalls as $k \to 0$.

To solve equation (9.65) for the critical dynamo wavenumber, it is helpful to view equation (9.65) as a *cubic* equation for $\zeta \equiv k_\star^2$. Unlike the sixth-order polynomial equation, the cubic is exactly solvable. Once we find the three (generally complex) values for ζ by standard means, we can take the square-root of each to obtain the six choices for $k_\star$. Based solely on the

[10] Why?

[11] Derive this result. Hint: Try to find a pair of PDE's for the y-components of $\langle \mathbf{A} \rangle$ and $\langle \mathbf{B} \rangle$.

[12] Verify that the second term on the RHS of equation (9.64) is indeed the square-root of the RHS of equation (9.63). This complex square-root formula is particularly useful result which is needed quite often in dealing with contour integration and complex variables. Notice that this definition of the complex square-root guarantees that the real part of the square-root is positive-definite, while the imaginary component can change its sign depending upon the sign of the product $\alpha\Omega\sin\vartheta$. It is also permitted to multiply the real part of the square-root by the factor $\mathrm{sign}(\alpha\Omega\sin\vartheta)$, instead of the imaginary part. This would guarantee that the imaginary part of the square-root is positive definite. In the parlance of complex analysis, choosing between either of these options is called picking a particular *Riemann sheet*.

coefficients of equation (9.65 it is possible to show that there is one real positive root, and a pair of complex-conjugate roots for the cubic ζ-equation. The lone positive root is the ticket, since (one) of its square-roots will also be positive and will provide us with the critical dynamo wavenumber that we seek. Rather than write out the result in all its detail, we will just remark that the critical dynamo wavenumber is readily estimated from equation (9.65 by inspection in the limiting cases:

$$k_\star \approx \begin{cases} \left[\frac{|\alpha\Omega\sin\vartheta|}{2\eta_e^2}\right]^{\frac{1}{3}} & \text{if } |\alpha| \ll \sqrt{\eta_e|\Omega\sin\vartheta|} \\ \frac{|\alpha|}{\eta_e} & \text{if } |\alpha| \gg \sqrt{\eta_e|\Omega\sin\vartheta|} \end{cases} \tag{9.66}$$

The upper line is generally thought to be most applicable to astrophysical situations, and the growing dynamo waves it predicts are called $\alpha\Omega$–dynamos. The lower line is associated with the α^2–dynamo wave.

We use the word "wave" to describe these exponentially growing solutions of the mean field equations because it is clear from equation (9.64) that $\text{Im}(\lambda_+) \neq 0$. The direction of propagation clearly depends upon the sign of the product of α and Ω, and the magnitude of the oscillation period is comparable to the growth rate for the $\alpha\Omega$–dynamo, but it is very much longer than this characteristic growth time for the α^2–dynamo wave. If we think about applying this simple Cartesian example to "explain" the solar cycle and the Maunder butterfly diagram, then our best bet is to hope that the $\alpha\Omega$–dynamo is in operation.

To conclude this section, let's see how well the $\alpha\Omega$–dynamo λ_+-solution that we found above will do in accounting for Figure 6.7. Before we plug in the numbers, we'll first get the geometry straight. The shear flow, you will recall, points in the $\hat{\mathbf{e}}_y$ direction, which we should associate locally with the $\hat{\mathbf{e}}_\phi$ direction in the spherical coordinate system. The $\alpha\Omega$–dynamo works best when the propagation direction of the dynamo wave is perpendicular *both* to the flow direction ($\hat{\mathbf{e}}_y$) and to the direction of shear ($\hat{\mathbf{e}}_z$). Therefore, to optimize our effort we should take $\vartheta = \pi/2$, so the dynamo wave propagates in the $\pm\hat{\mathbf{e}}_x$ direction in the Cartesian coordinate system, or equivalently the $\pm\hat{\mathbf{e}}_\theta$ on the Sun. So far so good. Using the right-hand-rule, this leaves $\hat{\mathbf{e}}_z$ corresponding to $\hat{\mathbf{e}}_r$. Hence, we have a radial shear of the mean zonal (azimuthal) flow (a.k.a. the differential rotation!), which in the presence of a non-zero α–effect, will lead to $\alpha\Omega$–dynamo waves propagating in the latitudinal direction. Excellent!

Now let's go back to the expression we have for λ_+ and put in the numbers. If τ is the assumed dynamo wave period, then, our requirement that we have a good working dynamo solution is,

$$|\alpha| = \frac{8\pi^2}{|\Omega|k\tau^2} \geq \frac{2\eta_e^2 k^3}{|\Omega|} \ . \tag{9.67}$$

The inequality guarantees that we have a growing dynamo wave solution, and the equality pegs its period to the observed value of τ. Following earlier discussions, we should place this dynamo wave on the tachocline between the solar envelope and the rigidly rotating solar radiative interior. This has the advantage of gaining us quite a hefty value for Ω, which in turn reduces the required efficiency of the α–effect.

Let's try to put some numbers on the RHS of eq. (9.67), in the specific case of the Sun, and more precisely. the equatorial base of the solar convection zone, where sunspot magnetic fields presumably originate. From the sunspot butterfly diagram we would guestimate $k \approx 4/(0.7R_\odot)$; estimates of convectrive velocities lead to $\eta_e \approx \beta \approx 10^{10}$ cm^2 s^{-1}, $\Omega \approx -130$ nHz from helioseismic inversions (cf. Fig. 7.5), and $\tau \approx 22$ yr. If you do the arithmetic, you find that we require $\alpha \approx +15$ cm s^{-1}—*positive* in order to get the dynamo wave to propagate from the pole *toward* the equator—and that we safely satisfy the required inequality by something like 5 orders of magnitude.

9.4 The mean-field dynamo equations

9.4.1 Axisymmetric formulation

We close this admittedly very mathematical chapter by getting back to the solar/stellar dynamo problem, and setting the stage for the following chapter, devoted to solar cycle models *per se*. Obviously, serious simplifications of the mean-field machinery is needed to yield as tractable problem. The stated goal, remember, is to produce models for the spatiotemporal evolution of the large-scale component of the magnetic field, while subsuming the inductive action of the small scale turbulent flow into the α- and β-effect terms of mean-field theory, as developed above. It is worth repeating that these are the two terms retained from a (severely) truncated series expansion of the mean electromotive force $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$ associated with the small-scale, fluctuating components of the velocity and magnetic field. You should also recall that the physical conditions under which this truncation can be expected to be meaningful may well not be satisfied under solar interior conditions, and that the rotationally-induced break of axisymmetry which allows to circumvent Cowling's theorem is completely contained in the α-effect.

We now proceed to reformulate the mean-field induction equation (9.16) into a form suitable for axisymmetric large-scale magnetic fields. We proceed as we did way back in §1.12.3), which is to express the poloidal field as the curl of a toroidal vector potential, and restrict the large-scale flow to the axisymmetric forms given by eq. (1.106). Henceforth dropping the averaging brackets for notational simplicity, the poloidal/toroidal separation procedure now leads to

$$\frac{\partial A}{\partial t} = \eta \left(\nabla^2 - \frac{1}{\varpi^2} \right) A - \frac{1}{\varpi} \mathbf{u}_p \cdot \nabla(\varpi A) + \alpha B \ , \tag{9.68}$$

$$\frac{\partial B}{\partial t} = \eta \left(\nabla^2 - \frac{1}{\varpi^2} \right) B - (\nabla \eta) \times (\nabla \times \mathbf{B})$$

$$-\varpi \nabla \cdot \left(\frac{B}{\varpi} \mathbf{u}_p \right) + \varpi (\nabla \times A) \cdot (\nabla \Omega) + \nabla \times [\alpha \nabla \times (A \hat{\mathbf{e}}_\phi)] \ , \tag{9.69}$$

which, structurally, only differs from eqs. (1.108)—(1.109) by the presence of two new terms on the RHS associated with the α-effect. The appearance of this term in eq. (9.68) is crucial, since this is allows us to evade Cowling's theorem.

Equations (9.68)–(9.69) will hereafter be refered to as the **dynamo equations** (rather than the technically preferable but cumbersome "axisymmetric mean-field dynamo equations"). For simplicity of notation, we continue to use η for the net magnetic diffusivity, with the understanding that this now includes the (presumably dominant) contribution from the β-term of mean-field theory.

In general, solutions are sought in a meridional plane of a sphere of radius R, and as with the diffusive problem of §7.1 are matched to a potential field in the exterior ($r/R > 1$). Regularity requires that the following boundary conditions be imposed on the symmetry axis:

$$A(r, 0) = A(r, \pi) = 0, \qquad B(r, 0) = B(r, \pi) = 0 \ . \tag{9.70}$$

In practice it is often useful to solve explicitly for mode having odd and even symmetry with respect to the equatorial plane. To do so, one simply solves the dynamo equations in a meridional *quadrant*, and imposes the following boundary conditions along the equatorial plane. For a dipole-like antisymmetric mode,

$$\frac{\partial A(r, \pi/2)}{\partial \theta} = 0, \qquad B(r, \pi/2) = 0 \ , \qquad [\text{Antisymmetric}] \ , \tag{9.71}$$

while for symmetric (quadrupole-like) modes one sets instead

$$A(r, \pi/2) = 0, \qquad \frac{\partial B(r, \pi/2)}{\partial \theta} = 0 \ , \qquad [\text{Symmetric}] \ . \tag{9.72}$$

9.4.2 Scalings and dynamo numbers

Our next step is to put the dynamo equations into nondimensional form. This can actually be carried out in a number of ways. We begin by scaling all lengths in terms of R, and time in terms of the diffusion time $\tau = R^2/\eta$. Equations (9.68)–(9.69) become

$$\frac{\partial A}{\partial t} = \left(\nabla^2 - \frac{1}{\varpi^2}\right) A - \frac{\mathrm{R}_m}{\varpi}\mathbf{u}_p \cdot \nabla(\varpi A) + C_\alpha \alpha B \ , \tag{9.73}$$

$$\frac{\partial B}{\partial t} = \left(\nabla^2 - \tfrac{1}{\varpi^2}\right) B - (\nabla\eta)\times(\nabla\times\mathbf{B}) - \mathrm{R}_m\varpi\nabla\cdot\left(\tfrac{B}{\varpi}\mathbf{u}_p\right)$$

$$+C_\Omega\varpi(\nabla\times A)\cdot(\nabla\Omega) + C_\alpha\nabla\times[\alpha\nabla\times(A\hat{\mathbf{e}}_\phi)] \ , \tag{9.74}$$

where the following three nondimensional numbers have materialized:

$$C_\alpha = \frac{\alpha_0 R}{\eta_0} \ , \tag{9.75}$$

$$C_\Omega = \frac{\Omega_0 R^2}{\eta_0} \ , \tag{9.76}$$

$$\mathrm{R}_m = \frac{u_0 R}{\eta_0} \ , \tag{9.77}$$

with α_0 (dimension m s^{-1}), η_0 (dimension m^2 s^{-1}), u_0 (dimension m^{2}s^{-1}) and Ω_0 (dimension s^{-1}) as reference values for the α-effect, diffusivity, meridional flow and shear, respectively. Remember that the functionals α, η, $\mathbf{u}_p$ and Ω are hereafter dimensionless quantities. The quantities C_α and C_Ω are **dynamo numbers**, measuring the importance of inductive versus diffusive effects on the RHS of eqs. (9.73)–(9.74). The third dimensionless number, R_m, is none other than our old friend the magnetic Reynolds number, which here measures the relative importance of advection (by meridional circulation) versus diffusion (by Ohmic dissipation) in the transport of A and B in meridional planes.

9.4.3 The little zoo of mean-field dynamo models

We now have a *two* source terms on the RHS of (9.74). As we will get to explore in subsequent chapters, whether or not one dominates over the other can lead to distinct modes of dynamo action.

Note first that dynamo action is now possible in the absence of a large-scale shear, i.e., with $\nabla\Omega = 0$ in eq. (9.74). Such dynamos are known as α^2 dynamos, and regenerate their magnetic field entirely via the inductive action of small-scale turbulence. Traditionally, dynamo action in planetary cores has been assumed to belong to this variety (at least from the point of view of mean-field theory).

Another possibility is that the shearing terms entirely dominates over the α-effect term, in which case the latter is altogether dropped out of eq. (9.74). This leads to the $\alpha\Omega$ dynamo model, which is believed to be most appropriate to the Sun and solar-type stars.

Finally, retaining both source terms in eq. (9.74) defines, you guessed it I hope, the $\alpha^2\Omega$ dynamo model. This has received comparatively little attention in the context of solar/stellar dynamos, since (simple) a priori estimates of the dynamo numbers C_α and C_Ω usually yield $C_\alpha/C_\Omega \ll 1$; caution is however warranted if dynamo action takes place in a thin shell...

Problems:

1. Carry out the averaging and separation procedure on the MHD induction equation, as described in §4.1, and show that it does lead to eqs. (9.4) and (9.5) for the mean and fluctuating parts of the magnetic field.
2. Show that under the scale separation assumption embodied in eqs. (9.2), the total magnetic energy in a (turbulent) volume V can be written as:

$$\mathcal{E}_B = \frac{1}{2\mu_0}\int_V \langle \mathbf{B} \rangle^2 \mathrm{d}V + \frac{1}{2\mu_0}\int_V \langle \mathbf{b}^2 \rangle \mathrm{d}V \ .$$

3. In the context of the plane-wave solutions discussed in §9.3, complete all missing mathematical steps leading to the dispersion relation given by eq. (9.63).

Bibliography:

The basic presentation of this chapter follows the excellent monograph,

Moffatt, H.K. 1978, *Magnetic Field Generation in Electrically Conducting Fluids*, (Cambridge: Cambridge Univ. Press).

Mean-field electrodynamics grew out of the original pioneering efforts of,

Parker, E.N. 1955, ApJ, 122, 293,
Braginskii, S.I. 1964, Sov. Phys. JETP, 20, 726; 1462,
Steenbeck, M., et al. 1966, Z. Naturforsch, 21a, 369; 1285.

Moffatt steers a middle course between the overtly mathematical and physical approaches to the problem. If your inclinations tend more toward the latter try,

Parker, E.N. 1979, *Cosmical Magnetic Fields*, (Oxford: Clarendon Press), ch. 18,

while those with an insatiable appetite for contracting tensors over various sets of indices will derive much satisfaction from perusing,

Krause, F., & Rädler, K.-H. 1980, *Mean-Field Magnetohydrodynamics and Dynamo Theory*, (Oxford: Pergamon Press).

A very reasonable introduction to correlation functions and attendant statistical concepts for random fields can be obtained from,

Batchelor, G.K. 1953, *The Theory of Homogeneous Turbulence*, (Cambridge: Cambridge Univ. Press).

It should me mentioned that to Batchelor, the appellation "homogeneous" also implies "mirror symmetric", which can lead to no small amount of confusion if you begin comparing some of our formulas with those in Batchelor's tome.

The turbulent mixing of a passive scalar is treated by,

Taylor, G.I. 1921, Proc. London Math. Soc., A20, 196,

while the potential for *negative* turbulent diffusion implied by equation (9.54) is discussed by

Kraichnan, R.H. 1976, J. Fluid Mech., 75, 657; 77, 753,
Parker, E.N. 1979, *Cosmical Magnetic Fields*, (Oxford: Clarendon Press), pp. 584-592.

On empirical estimates of the α-effect from numerical simulations of MHD turbulence, start with:

Pouquet, A., Frisch, U., and Lorat, J. 1976, *J. Fluid Mech.*, **77**, 321,
Nordlund, Å, Brandenburg, A., Jennings, R.L., Rieutord, M., Ruokalaien, J., Stein, R.F., & Tuominen, I. 1992, *Astrophys. J.*, **392**, 647,
Ossendrijver, M., Stix, M., and Brandenburg, A. 2001, *Astron. Ap.*, **376**, 731.