nozzle's cross-section. Now, in a polytropic flow, we already saw that the Bernoulli constant, corresponding to the total energy per unit mass, is given by:

$$E = \frac{u_r^2}{2} + \frac{c_s^2}{\alpha - 1} - \frac{GM}{r} \ . \tag{4.11}$$

You'll recall (hopefully) that the three terms on the RHS are, from left to right: the flow's kinetic energy, the plasma's thermal energy, and gravitational potential energy, all per unit mass. The most any nozzle can do, starting from a fluid at rest in the "combustion chamber" (here the coronal base) is to convert all of the plasma's original thermal energy into outflow kinetic energy; here this limiting velocity is given by something like:

$$u_\infty = \frac{2c_{s0}^2}{\alpha - 1} - \frac{GM}{r_0} \ , \tag{4.12}$$

where as before c_{s0} is the sound speed at r_0. But how does this work out in practice? Working once again through the mathematical steps we encountered in the case of Parker's spherically symmetric polytropic wind solution, it can be shown that for an arbitrary expansion factor $A(r)$, the r-momentum equation can be written in the following general form:

$$\frac{M^2-1}{2M^2}\frac{\mathrm{d}M^2}{\mathrm{d}r} = \left[1 + \left(\frac{\alpha-1}{2}\right)M^2\right]\left[\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r} - \frac{1}{2}\left(\frac{\alpha+1}{\alpha-1}\right)\frac{GM/r^2}{(E+GM/r)}\right]$$

$$= \frac{1}{2}\left(\frac{\alpha+1}{\alpha-1}\right)\left[1 + \left(\frac{\alpha-1}{2}\right)\right]\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}r} \ , \tag{4.13}$$

where E is the Bernoulli constant of eq. (4.11), M is the Mach number

$$M(r) = \frac{u_r}{c_s} \ , \tag{4.14}$$

and the function g is given by

$$g(r) = A^{2(\alpha-1)/(\alpha+1)}\left(E + \frac{GM}{r}\right) \ . \tag{4.15}$$

The mathematics are more complex, but this is really the same general idea as with the spherically-symmetric Parker polytropic wind solution of §3.3. In particular, solutions to eq. (4.13) include sonic critical points that must be crossed by wind solution in order to avoid infinite accelerations. What is novel here is that for expansion factors with fast divergence, more than one critical points can exist in the flow. You get to explore this aspect of the problem in Problem 4.5 below.

An integral form of eq. (4.13) can also be obtained:

$$M^{4/(\alpha+1)} + \left(\frac{2}{\alpha-1}\right)M^{-2(\alpha-1)/(\alpha+1)}$$

$$= \frac{g(r)}{g_0}\left[M_0^{4/(\alpha+1)} + \left(\frac{2}{\alpha-1}\right)\right]M_0^{-2(\alpha-1)/(\alpha+1)} \ , \tag{4.16}$$

with $g_0 \equiv g(r_0)$ and $M_0 \equiv M(r_0)$. This is really nothing more than the Bernoulli equation (4.11) written in terms of the Mach number. This form is useful for reconstructing full wind solution, since it amounts to yet another root-finding problem for M (at fixed r).

4.6 The $\beta \gg 1$ limit: The Parker spiral

Consider now the other opposite, extreme limit of $\beta \gg 1$, in which the magnetic field is passively advected by the wind outflow. More specifically, assume a steady-state situation where

1. Flux-freezing is effectively enforced,
2. Magnetic stresses are neglected in the force balance,
3. The poloidal part of the magnetic field is purely radial in the equatorial plane, with the field strength known at the reference radius.

In view of (3), the constraint $\nabla \cdot \mathbf{B} = 0$ is readily integrated to

$$B_r(r) = B_{r0} \left(\frac{r_0}{r} \right)^2 . \tag{4.17}$$

For an average surface field $B_0 \sim 10^{-3}\,$T, eq. (4.17) yields $B_r \simeq 25\,$nT at the Earth's orbit, which is not that far from the observed average magnetic field at 1AU. In view of (1), the flow streamlines coincide with magnetic fieldlines. In the absence of rotation, the Parker solution is immediately applicable. Consider now the introduction of rotation, at a rate Ω such that centrifugal effects do not affect significantly the force balance in the r-direction. In a frame co-rotating with the Sun, the wind still flows along the magnetic fieldlines. But in a stationary frame, the total velocity is now

$$\mathbf{u} = \mathbf{u}' + \Omega\, r\, \hat{\mathbf{e}}_\phi , \tag{4.18}$$

where primed quantities refer to quantities evaluated in the co-rotating frame. In general, for a constant-speed radial outflow the magnetic fieldlines is defined by the spiral

$$r = (u_r/\Omega_\odot)(\phi - \phi_0) \ , \tag{4.19}$$

with the r and ϕ-components of the magnetic field given by

$$B_r(r,\theta,\phi) = B_r(r_0,\theta,\phi - r\Omega_\odot/u_r) \left(\frac{r_0}{r} \right)^2 , \tag{4.20}$$

$$B_\phi(r,\theta,\phi) = B_r(r_0,\theta,\phi - r\Omega_\odot/u_r) \left(\frac{r_0\Omega_\odot}{u_r} \right) \left(\frac{r_0}{r} \right) . \tag{4.21}$$

Figure 4.4 shows, in the equatorial plane, the magnetic fieldlines defined by eqs. (4.20)–(4.21) with $u_r = 350\,\mathrm{km\ s}^{-1}$ and the dashed circle corresponding to the Earth's orbit[3]. The angle between a magnetic fieldline and the Sun-Earth radial segment is:

$$\phi_B = \arctan \left(\frac{B_\phi}{B_r} \right) = \arctan \left(\frac{r\Omega_\odot}{u_r} \right) , \tag{4.22}$$

which at 1 AU gives the rather large value $\phi_B \simeq 55^o$, which in fact compares favorably with observations. The net wind velocity at 1 AU, on the other hand, is essentially radial[4].

Now, remember the equatorial current sheet that characterized the partially open magnetostatic solutions considered in §4.3? Well this has been detected also through situ solar wind observations at 1AU. One of the most intriguing aspect of early space-borne solar wind measurements was the semi-regular polarity flips of the magnetic field carried by the wind. It was soon realized that this could be traced to the fact that the "neutral line" $B_r = 0$ at the solar surface does not coincide exactly with the equatorial circle, but is often deformed into a wavy

[3] On Fig. 4.4, is the Sun rotating clockwise or counterclockwise?

[4] Confusion on the horizon. Didn't we argue that in the flux-freezing limit, the gas could only flow parallel to the fieldlines? Shouldn't we then have $\arctan(u_\phi/u_r) \simeq 55^o$ also? How do you explain this?

so that

$$B_\phi = \frac{B_r}{u_r}(u_\phi - \Omega r) \ . \tag{5.13}$$

Now, eq. (5.4) can obviously be rewritten as

$$\frac{\partial}{\partial r}(r u_\phi) = \frac{B_r}{\mu_0 \rho u_r}\frac{\partial}{\partial r}(r B_\phi) \ ; \tag{5.14}$$

but in view of eqs. (5.7) and (5.8), we have $B_r/\mu_0\rho u_r = C_2/\mu_0 C_1$, i.e., a constant! Which means that eq. (5.14) integrates immediately to

$$r u_\phi - \frac{r B_\phi B_r}{\mu_0 \rho u_r} = L \ , \tag{5.15}$$

where L is yet another integration constant. It has a well-defined physical meaning, as it corresponds to the total angular momentum carried away by the wind, which is made up of two contributions: the specific angular momentum of the expanding fluid (first term on LHS), and the torque density associated with magnetic tension (remember that the magnetic field is being dragged away by the wind outflow!)

The results of all this algebraic juggling, without giving us a full wind solution, still allow us to draw some interesting conclusions regarding the behavior of the outflow. First we rewrite eqs. (5.13) and eqs. (5.15) in terms of the components of the Alfvén velocity[1] (§1.8):

$$A_r = \frac{B_r}{\sqrt{\mu_0 \rho}} \ , \qquad A_\phi = \frac{B_\phi}{\sqrt{\mu_0 \rho}} \ , \tag{5.16}$$

leading to

$$A_\phi = \frac{A_r}{u_r}(u_\phi - \Omega r) \ , \tag{5.17}$$

$$u_\phi = \frac{L}{r} + \frac{A_r A_\phi}{u_r} \ . \tag{5.18}$$

Substituting now for A_ϕ in eq. (5.18) and making good use of eqs. (5.16) and eqs. (5.17) yield, after some straightforward algebra:

$$u_\phi = \Omega r \frac{(u_r^2 L/\Omega r^2) - A_r^2}{u_r^2 - A_r^2} \ . \tag{5.19}$$

Look at the denominator of this expression; clearly, if the radial flow velocity ever becomes equal to the radial Alfvén speed, we are in trouble... unless the numerator also happens to vanish. We can save the day in this way provided we set

$$\boxed{L = \Omega r_A^2} \ , \tag{5.20}$$

where r_A is the **Alfvén radius**, defining the spherical shell where $u_r = A_r$. Now, remember that L is the total angular momentum carried away by the wind, *including* the torque density provided by magnetic tension. Equation (5.20) states that this is equal to the angular momentum that would be carried away by an unmagnetized wind flowing strictly radially, and co-rotating with the solar/stellar surface out to radius r_A. We are going to get a lot of mileage from this remarkable result later on. But let's first try to get a full wind solution. Go back to the r-component of the equation of motion (eq. (5.3)); use eq. (5.13) to eliminate B_ϕ in the

[1]Please do not confuse the "A" here with components of the magnetic vector potential...

last term on the RHS; then use eq. (5.13) to eliminate the B_ϕ derivative multiplying u_ϕ (but leave the one multiplying Ω alone!). Somewhat tedious algebra eventually leads to

$$\frac{\partial}{\partial r}\left[\frac{1}{2}(u_r^2+u_\phi^2)-\frac{GM}{r}+\frac{c_{s0}^2}{\alpha-1}\left(\frac{\rho}{\rho_0}\right)^{\alpha-1}-\frac{r\Omega A_r A_\phi}{u_r}\right]=0\ , \tag{5.21}$$

where the magnetic field components are again expressed in terms of their corresponding Alfvén speed components, and the polytropic approximation was used to deal with the pressure gradient term. This indicates that the quantity within the square brackets must be a constant[2]. This is again a Bernoulli-type statement for the flow, expressing conservation of energy, and as before we will denote the quantity in square brackets by E.

Obtaining a full solution (i.e., u_r, $u_\phi(r)$, etc.) is now a much more complicated procedure. The starting point is the manipulation of eq. (5.3) into the form:

$$\frac{\partial u_r}{\partial r}=\left(\frac{u_r}{r}\right)\frac{(u_r^2-A_r^2)(2c_s^2+u_\phi^2-GM/r)+2u_r u_\phi A_r A_\phi}{(u_r^2-A_r^2)(u_r^2-c_s^2)-u_r^2A_\phi^2}\ , \tag{5.22}$$

which involves some straightforward but tedious algebraic juggling. Now, that denominator looks like trouble once again. It actually vanishes whenever the radial flow speed u_r becomes equal to the phase speed of either the slow or fast magnetosonic wave modes[3], which in general occurs at distinct radial distances denoted r_s and r_f in what follows. Denote now by N and D the numerator and denominator on the RHS of eq. (5.22); to avoid divergence at r_s or r_f we require that

$$N(r_f,u_f)=0\ , \tag{5.23}$$
$$D(r_f,u_f)=0\ , \tag{5.24}$$
$$N(r_s,u_s)=0\ , \tag{5.25}$$
$$D(r_s,u_s)=0\ , \tag{5.26}$$

complemented by the requirement that solutions running through the two critical points should also be characterized by the same value of the Bernoulli constant[4]:

$$E(r_f,u_f)=E(r_0,u_{r0})\ , \tag{5.27}$$
$$E(r_s,u_s)=E(r_0,u_{r0})\ . \tag{5.28}$$

These expressions represent a set of six coupled nonlinear algebraic equations that must be solved simultaneously for a "solution vector"

$$\mathbf{w}=(u_{r0},u_{\phi0},r_s,u_s,r_f,u_f)\ . \tag{5.29}$$

Well, we can find reassurance in the fact that we have as many equations as we have unknowns, but the fact remains that solving this nonlinear algebraic system is A BEAR of a root finding problem. It can be turned into a (somewhat easier) optimization problem, by seeking solutions that minimize the sum of the squared N's, D's and E's, but even then you better have a pretty good initial guess for the solution vector to start a conjugate gradient or whatever, because the

[2]If eq. (5.21) doesn't look at least a bit familiar, go back and read chapter 3, before proceeding, because you're already in trouble enough.

[3]Remember that these correspond to sound-like longitudinal waves for which the sum of gas and magnetic pressures act as a restoring force; if both are in (out of) phase, the magnetosonic wave is fast (slow). If you don't remember, `goto` §1.8, do not pass GO, do not collect $200

[4]Hold on now, didn't we say a little while back that the wind also had to go through the Alfvén point, to avoid a blowup of the azimuthal velocity, as per eq. (5.19)? Well it turns out that in the Weber-Davis-type wind models, any solution going through the slow and fast magnetosonic points (r_s,u_s), (r_f,u_f) *automatically* goes through the Alfvén point (r_A,u_{rA}). Skeptics should either get a life, or consult Goldreich & Julian 1970, ApJ, 160, 971.

which readily integrates to

$$\frac{1}{\Omega^2(t)} - \frac{1}{\Omega^2(t_0)} \propto t - t_0 \ , \tag{5.43}$$

where t_0 is the time of arrival on the ZAMS (or shortly thereabouts). In the asymptotic limit $t \gg t_0$, $\Omega \ll \Omega(t_0)$, this becomes

$$\Omega(t) \propto t^{-1/2} \ , \tag{5.44}$$

which, how about that, is precisely the power-law relationship inferred observationally by Skumanich (cf. Fig. 5.7). Looks like we're in business!

5.3.3 The spindown of late-type stars

The missing proportionality constant in eq. (5.44) is of course readily computed from our Weber-Davis solution; in fact we did nearly all the work already in arriving at eq. (5.41), the missing element being the expression of stellar angular momentum in terms of a star's angular velocity distribution. If, for the time being, we assume that stars rotate as rigid bodies, then we have

$$J = I_* \Omega_* \ , \tag{5.45}$$

and dimensional analysis yields the following expression for the **spin-down timescale**:

$$\tau_{\rm sp} = I_* \Omega_* \left(\frac{{\rm d}J}{{\rm d}t} \right)^{-1} \ . \tag{5.46}$$

All we are missing are the stellar moments of inertia, which are readily computed if we have stellar structural models on hand. The third column of Table 5.4 list the resulting spin-down timecales, for ZAMS stellar models between 0.8 and 1.2 $M_\odot$. In all cases it as assumed that the ZAMS rotation period is one day, and the radial surface magnetic field strength is 50 G, reasonable numbers to the extent we can tell from observations and models of stellar formation and pre-main-sequence evolution.

Table 5.4
ZAMS spindown timescales for late-type stars[6]

$M/M_\odot$	$R/R_\odot$	$I_*[10^{53}]$	$\tau_{J,*}$ [Myr]	$I_E[10^{53}]$	$\tau_{J,E}$ [Myr]
0.8	0.703	4.41	810	1.025	188
0.9	0.784	5.50	604	0.979	107
1.0	0.882	6.75	396	0.833	48.9
1.2	1.131	9.02	133	0.139	2.05

The spin-down timescales are of order 10^8 and 10^9 yr, which is nicely smaller than the solar age, but a factor of ten longer than the spin-down timecales inferred from $v \sin i$ determinations in young stellar clusters. Observations do offer an interesting hint, in that after arriving on the main-sequence, more massive stars seem to spin down *faster* than less massive stars, even though their moment of inertia is larger (second column of Table 5.4).

The favored escape from this quandary is to assume that the torque applied by the wind to the photospheric layers is not transmitted throughout the whole star, but (at first anyway) only to its convective envelope, where the vigorous turbulent thermally-driven convective fluid motions are expected to redistribute momentum on the convective turnover time, of the order of a month for convection in solar-type stars. Now, the thickness of the convection decreases

[6]Stellar structural models courtesy of S. Vanderberg, U. Victoria.

rapidly as mass increases, leading to a decrease of the moment of inertia of main-sequence convective envelope with increasing mass (see fifth column in 5.4. This then leads to spin-dowm times (last column in Table 5.4) that (1) are in much better agreement with observationally-inferred values (2) *decrease* with increasing mass. It all fits together!

In late type stars spun down by a wind-mediated surface torque, many physical processes can exchange angular momentum between the convective envelope and underlying radiative core. Indeed, helioseismology has shown that the angular velocity of the solar core is comparable to that of its convective envelope, implying that whatever dynamical coupling is taking place between the core and envelope acts on a timescale much smaller than the solar age (but still significantly longer that the ZAMS spindown timescales, otherwise we're in trouble again). It turns out that internal magnetic fields can do the trick, and remain at this writing the most physically viable explanation for the rotation rate of the solar radiative core. To substantiate this claim would take us too deep inside the sun, but references listed in the bibliography to this chapter provide good entry points into this area of research. Time to get back up into the wind and see what we can do about those famous high-speed streams...

5.4 Wind driving by Alfvén waves

In the solar photosphere, the plasma-β is high enough that magnetic fieldlines get continuously displaced by convective fluid motions. Vertical displacements will generate magnetosonic waves, which are expected to shock and dissipate before they reach the corona. Horizontal displacements of magnetic fieldlines, on the other hand, will propagate upwards into the corona in the form of Alfvén waves. These, it turns out, can have a significant dynamical impact on wind-like outflows, and this is what we'll look into in this section.

The physical/geometrical setup we consider here closely resembles that of the Weber-Davis solution of §5.1, i.e., working in spherical polar coordinates we solve the steady ($\partial/\partial t = 0$) axisymmetric ($\partial/\partial\phi = 0$) wind equations in the equatorial plane of the star, assuming a radial reference magnetic field. The two important differences are: (1) rotation is neglected, and (2) we consider an isothermal, rather than polytropic wind, otherwise the mathematics really get too messy.

The key in formulating the wave-wind model is to assume that the total flow and magnetic field can be written as

$$\mathbf{u}(r,t) = u_r(r)\hat{\mathbf{e}}_r + \delta u(r,t)\hat{\mathbf{e}}_\phi \ , \tag{5.47}$$

$$\mathbf{B}(r,t) = B_r(r)\hat{\mathbf{e}}_r + \delta B(r,t)\hat{\mathbf{e}}_\phi \ , \tag{5.48}$$

where u_r, B_r define the large-scale wind outflow, and the two leftmost terms correspond to a transverse wave travelling in the r-direction and "oscillating" in the ϕ-direction; that latter choice is entirely arbitrary, but will facilitate the mathematical developments to follow. As with any wave, the time averages of the local wave contribution to the flow and field vanish:

$$\langle \delta u \rangle = 0 \ , \qquad \langle \delta B \rangle = 0 \ . \tag{5.49}$$

5.4.1 The magnetic force exerted by Alfvén waves

Looking at the momentum equation, you should be able to convince yourself that the contribution to the r-component of the force per unit volume (f_w) associated with the wave is given by:

$$f_w = \left(\rho(\delta\mathbf{u}\cdot\nabla)\delta\mathbf{u} + \frac{1}{\mu_0}(\nabla\times\delta\mathbf{B})\times\delta\mathbf{B} \right)_r \ . \tag{5.50}$$