PHY-6795

MAGNETOHYDRODYNAMIQUE ASTROPHYSIQUE

Notes de cours

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AVERTISSEMENT

Les notes qui suivent ont été originellement préparées dans le cadre d'un cours gradué en Physique Solaire enseigné à trois reprises par Tom Bogdan et moi-même à l'Université du Colorado à Boulder, sous les sigles APAS7500 et ASTR7500, ainsi qu'en automne 2004 à l'UdeM dans la cadre de la première moitié de PHY6795A. Vous remarquerez rapidement que ces notes de cours sont écrites en anglais. La raison principale en est que ce que vous avez en main représente une partie d'un ouvrage gradué sur la magnétohydrodynamique solaire, que Tom et moi espérons bien finir par publier un de ces jours... Comme vous le constaterez rapidement, tous les chapitres n'en sont pas au même stade de "perfectionnement"; Tom et moi apprécierions grandement tout commentaires et suggestions que vous pourriez nous faire quant au contenu et à la présentation des sujets couverts ici.

La magnétohydrodynamique astrophysique est un sujet particulièrement vaste; fidèle à l'approche adoptée dès la première version de ce cours, J'ai choisi ici de couvrir deux sujets en détails, plutôt que de faire un survol inévitablement superficiel de tout le domaine.

En électromagnétisme, le choix d'unités a des conséquences non-triviales, entre autre au niveau de la forme que prennent les équations de Maxwell. Bien que les unités CGS demeurent la norme en astrophysique, j'ai finalement choisi d'utiliser dans ces notes le système SI (alias MKS). J'ai cependant inclut un petit Appendice qui, je l'espère, devrait aider les accros du système CGS à s'y retrouver dans le passage des gauss aux tesla, des weber aux maxwell, etc.

Au fil des années et à travers les diverses incarnations de ces notes de cours, plusieurs collègues ont apporté des contributions importantes et/ou gracieusement fourni des diagrammes ou résultats de calculs numériques servant à illustrer certains aspects de la matière couverte. Je tiens particulièrement à remercier Fausto Cattaneo, David Galloway, Keith MacGregor, Steve Tobias, Mathieu Ossendrijver, Nic Brummell, Rony Keppens et Gregg Wade. Merci également à Jacques Richer pour sa précieuse aide avec les subtilités du LaTeX avancé

La version 2008 de ces notes a grandement bénéficié d'une lecture critique en août 2008 par Michel-André Vallières-Nollet; en plus de lui offrir ici mes plus chaleureux remerciements, je profite de cette pseudo-préface pour lui promettre une lecture tout aussi attentive de son mémoire de maitrise!

Paul Charbonneau
Montréal, septembre 2008

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Part I

Introduction

Chapter 1

Magnetohydrodynamics

From a long view of history —seen from, say, ten thousand years from now— there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.

Richard Feynman
The Feynmann Lectures on Physics (1964)

To sum it all up in a single sentence, **magnetohydrodynamics** (hereafter MHD) is concerned with the behavior of electrically conducting but globally neutral fluids flowing at non-relativistic speeds and obeying Ohm's Law. Before we dive into MHD proper, it would be wise to clarify what we mean by "fluid" (§1.1), and review the fundamental physical laws governing the flow of unmagnetized fluid, i.e., classical hydrodynamics (§1.2). We then introduce magnetic fields into the fluid picture (§§1.3—1.11), and close with useful mathematico-physical tidbits.

1.1 The fluid approximation

1.1.1 Matter as a continuum

It did take some two thousand years to figure it out, but we now know that Democritus was right after all: matter is composed of small, microscopic "atomic" constituents. Yet on our daily macroscopic scale, things sure look smooth and continuous. Under what circumstances can an assemblage of microscopic elements be treated as a continuum? The key constraint is that there be a good **separation for scales** between the "microscopic" and "macroscopic".

Consider the situation depicted on Figure 1.1, corresponding to an amorphous substance (spatially random distribution of microscopic constituents). Denote by λ the mean interparticle distance, and by L the macroscopic scale of the system; we now seek to construct macroscopic variables defining fluid characteristics at the macroscopic scale. For example, if we are dealing with an assemblage of particles of mass m, then the **density** (ρ) associated with a cartesian volume element of linear dimensions l centered at position $\mathbf{x}$ would be given by something like:

$$\rho(\mathbf{x}) = \frac{1}{l^3}\sum_k m_k \ , \qquad [\mathrm{kg\,m^{-3}}] \ , \tag{1.1}$$

where the sum runs over all particles contained within the volume element. One often hears or reads that for a continuum representation to hold, it is only necessary that the density be "large". But large with respect to what? For the above expression to yield a well-defined quantity, in the sense that the numerical value of ρ so computed does not depend sensitively on the size and location of the volume element, or on time if the particles are moving, it is essential that a great many particles be contained within the element. Moreover, if we want

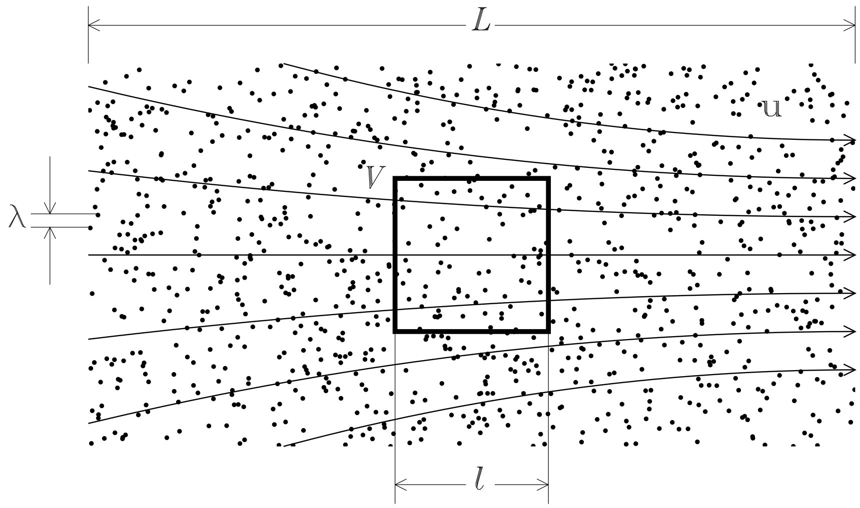

Figure 1.1: Microscopic view of a fluid. In general the velocity of microscopic constituents is comprised of two parts: a randomly-oriented thermal velocity, and a systematic drift velocity, which, on the macroscopic scale amounts to what we call a flow **u**. A fluid representation is possible if the mean inter-particle distance λ is much smaller than the global length scale L.

to be writing differential equations describing the evolution of ρ, the volume element better be infinitesimal, in the sense that it is much smaller that the macroscopic length scale over which global variables such as ρ may vary. These two requirements translate in the double inequality:

$$\lambda \ll l \ll L \ . \tag{1.2}$$

Because the astrophysical systems and flows that will be the focus of our attention span a very wide range of macroscopic sizes, the continuum/fluid representation will turn out to hold in circumstances where the density is in fact minuscule, as you can verify for yourself upon perusing the collection of astrophysical systems listed in Table 1.1 below[1]. In all cases, a very good separation of scales does exist between the microscopic (λ) and macroscopic (L).

Table 1.1
Spatial scales of some astrophysical objects and flows

System/flow	ρ [kg/m^3]	N [m^{-3}]	λ [m]	L [km]
Solar interior	100	10^{29}	10^{-10}	10^5
Solar atmosphere	10^{-4}	10^{23}	10^{-8}	10^3
Solar corona	10^{-11}	10^{17}	10^{-6}	10^5
Solar wind (1 AU)	10^{-21}	10^7	0.006	10^5
Molecular cloud	10^{-20}	10^7	0.001	10^{14}
Interstellar medium	10^{-21}	10^6	0.01	10^{16}

[1]All density-related estimate assume a gas of fully ionized Hydrogen ($\mu = 0.5$) for the Sun, of neutral Hydrogen for the interstellar medium ($\mu = 1$), and molecular Hydrogen ($\mu = 2$) for molecular clouds. Solar densities are for the base of the convection zone (solar interior), optical depth unity (atmosphere), and typical coronal loop (corona). N is the number density of microscopic constituents. The length scale listed for the solar atmosphere is the granulation dimension, for the corona it is the length of a coronal loop, for the solar wind the size of Earth's magnetosphere, and that for the interstellar medium is the thickness of the galactic (stellar) disk; All rounded to the nearest factor of ten.

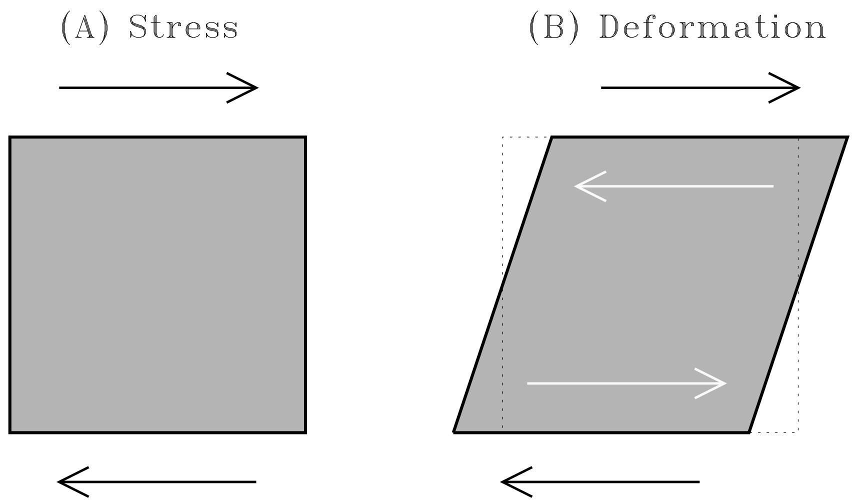

Figure 1.2: Deformation of a mass element in response to a stress pattern producing an horizontal shear (black arrows). A solid will rapidly reach an equilibrium where internal stresses (white arrows) produced by the deformation will equilibrate the applied shear. A fluid at rest cannot generate internal stresses, and so will be increasingly deformed for as long as the external shear is applied.

1.1.2 Solid versus fluid

Most continuous media can be divided into two broad categories, namely **solids** and **fluids**. The latter does not just include the usual "liquids" of the vernacular, but also gases and plasmas. Physically, the distinction is made on the basis of a medium's response to an applied **stress**, as illustrated on Figure 1.2. A volume element of some continuous substance is subjected to a shear stress, i.e., two force acting tangentially and in opposite directions on two of its parallel bounding surface (black arrows). A **solid** will immediately generate a restoring force (white arrows), ultimately due to electrostatic interactions between its microscopic constituents, and vigorously resist deformation (try shearing a brick held between the palms of your hands!). The solid will rapidly reach a new equilibrium state characterized by a finite deformation, and will relax equally rapidly to its initial state once the external stress vanishes. A **fluid**, on the other hand, can offer no resistance to the applied stress, at least in the initial stages of the deformation[2].

1.2 Essentials of hydrodynamics

The governing principles of classical hydrodynamics are the same as those of classical mechanics, transposed to continuous media: conservation of mass, linear momentum, angular momentum and energy. The fact that these principles must now be applied not to point-particles, but to spatially extended volume elements (which may well be infinitesimal, but they are still finite!) introduces some significant complications, mostly with regards to the manner in which forces act. Let's start with the easiest of our conservation statements, that for mass, as it exemplifies very well the manner in which conservation laws are formulated in moving fluids.

[2]We will return in due time to what happens once contiguous fluid elements have attained different, finite velocities. In short, the restoring force is often proportional to the velocity **gradient** produced by the action of the shear.

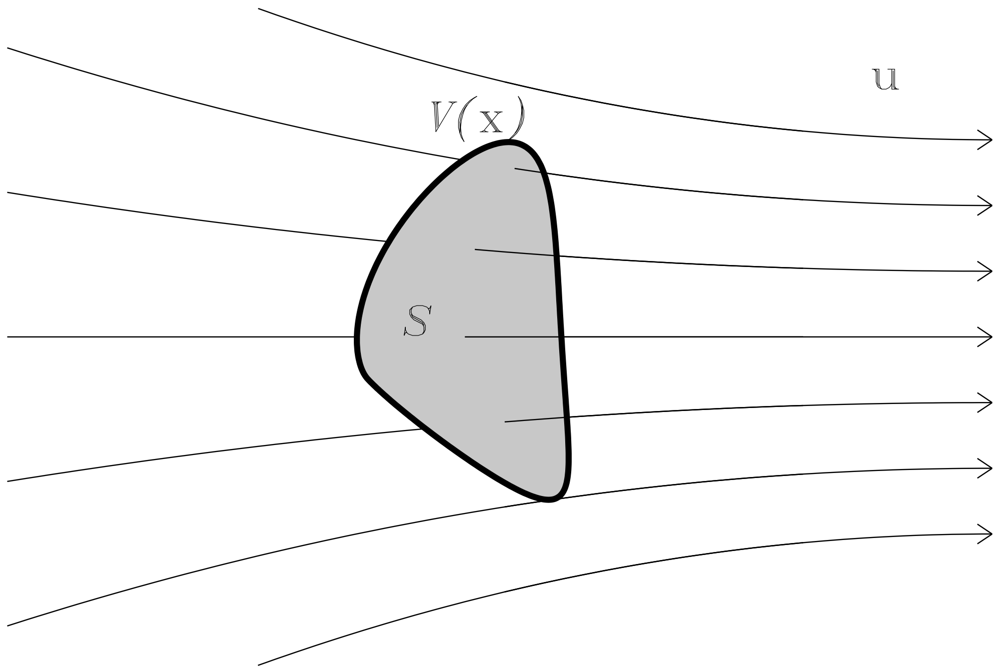

Figure 1.3: An arbitrarily shaped volume element V bounded by a closed surface S, both fixed in space, and traversed by a flow $\mathbf{u}$.

1.2.1 Mass: the continuity equation

Consider the situation depicted on Figure 1.3, namely that of an arbitrarily shaped surface S fixed in space and enclosing a volume V embedded in a fluid of density $\rho(\mathbf{x})$ moving with velocity $\mathbf{u}(\mathbf{x})$. The **mass flux** associated with the flow across the (closed) surface is

$$\Phi = \oint_S \rho \mathbf{u} \cdot \hat{\mathbf{n}} \mathrm{d}S \ , \qquad [\mathrm{kg\,s^{-1}}] \tag{1.3}$$

where $\hat{\mathbf{n}}$ is a unit vector everywhere perpendicular to the surface, and by convention oriented towards the exterior. The mass of fluid contained within V is simply

$$M = \int_V \rho \, \mathrm{d}V \ . \qquad [\mathrm{kg}] \tag{1.4}$$

This quantity will evidently vary if the mass flux given by eq. (1.3) is non-zero:

$$\frac{\partial M}{\partial t} = -\Phi \ . \tag{1.5}$$

Here the minus sign is a direct consequence of the exterior orientation of $\hat{\mathbf{n}}$. Inserting eq. (1.3) and eq. (1.4) into (1.5) and applying the divergence theorem to the RHS of the resulting expression yields:

$$\frac{\partial}{\partial t} \int_V \rho \mathrm{d}V = -\int_V \nabla \cdot (\rho \mathbf{u}) \mathrm{d}V \ . \tag{1.6}$$

Because V is fixed in space, the $\partial / \partial t$ et $\int_V$ operators commute, so that

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \mathrm{d}V = 0 \ . \tag{1.7}$$

Because V is completely arbitrary, in general this can only be satisfied provided that

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0} \ . \tag{1.8}$$

This expresses mass conservation in differential form, and is known in hydrodynamics as the **continuity equation**.

Incompressible fluids have constant densities, so that in this limiting case the continuity equation reduces to

$$\nabla \cdot \mathbf{u} = 0 \ , \qquad \text{[incompressible]}. \tag{1.9}$$

Water is perhaps the most common example of an effectively incompressible fluid (under the vast majority of naturally occuring conditions anyway). The gaseous nature of most astrophysical fluids may lead you to think that incompressibility is likely to be a pretty lousy approximation in cases of interest in this course. It turns out that the incompressible approximation can lead to a pretty good approximation of the behavior of compressible fluids provided that the flow's Mach number (ratio of flow speed to sound speed) is much smaller than unity.

1.2.2 The D/Dt operator

Suppose we want to compute the time variation of some physical quantity (Z, say) at some fixed location $\mathbf{x}_0$ in a flow $\mathbf{u}(\mathbf{x})$. In doing so we must take into account the fact that Z is in general both an explicit and implicit function of time, because the volume element "containing" Z is moving with the fluid, i.e., $Z \to Z(t, \mathbf{x}(t))$. We therefore need to use the chain rule and write:

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial Z}{\partial z}\frac{\partial z}{\partial t} \ . \tag{1.10}$$

Noting that $\mathbf{u} = \mathrm{d}\mathbf{x}/\mathrm{d}t$, this becomes

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial x}u_x + \frac{\partial Z}{\partial y}u_y + \frac{\partial Z}{\partial z}u_z = \frac{\partial Z}{\partial t} + (\mathbf{u} \cdot \nabla)Z \ . \tag{1.11}$$

This corresponds to the time variation of Z *following the fluid element as it is carried by the flow*. It is a very special kind of derivative in hydrodynamics, known as the **Lagrangian derivative**, which will be represented by the operator:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)} \ . \tag{1.12}$$

Note in particular that the Lagrangian derivative of $\mathbf{u}$ yields the acceleration of a fluid element:

$$\mathbf{a} = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \ , \tag{1.13}$$

a notion that will soon come very handy when we'll write $F = ma$ for a fluid.

A **material surface** is defined as an ensemble of points that define a surface, all moving along with the flow. Therefore, in a local frame of reference S' co-moving with any infinitesimal element of a material surface, $\mathbf{u}' = 0$. The distinction between material surfaces, as opposed to surfaces fixed in space such as in eq. (1.3), has crucial consequences with respect to the commuting properties of temporal and spatial differential operators. In the latter case $\int_V$ commutes with $\partial/\partial t$, whereas for material surfaces and volume elements it is D/Dt that commutes with $\int_V$ (and $\oint_S$, etc.).

1.2.3 Linear momentum: the Navier-Stokes equations

A force $\mathbf{F}$ acting on a point-object of mass m is easy to deal with; it simply procuces an acceleration $\mathbf{a} = \mathbf{F}/m$ in the same direction as the force (sounds simple but it still took the genius of Newton to figure it out...). In the presence of a force acting on the surface of a spatially extended fluid element, the resulting fluid acceleration will depend on both the orientation of the force and the surface. We therefore define the net force $\mathbf{t}$ in terms of a **stress tensor**:

$$\mathbf{t}_x = \hat{\mathbf{e}}_x s_{xx} + \hat{\mathbf{e}}_y s_{xy} + \hat{\mathbf{e}}_z s_{xz} \ , \qquad (1.14)$$

$$\mathbf{t}_y = \hat{\mathbf{e}}_x s_{yx} + \hat{\mathbf{e}}_y s_{yy} + \hat{\mathbf{e}}_z s_{yz} \ , \qquad (1.15)$$

$$\mathbf{t}_z = \hat{\mathbf{e}}_x s_{zx} + \hat{\mathbf{e}}_y s_{zy} + \hat{\mathbf{e}}_z s_{zz} \ , \qquad (1.16)$$

where "s_{xy}" denotes the force per unit area acting in the y-direction on a surface perpendicular to the x-direction, $\mathbf{t}_x$ is the net force acting on the surfaces perpendicular to the x-direction, and similarly for the other components. Consider now a unit vector perpendicular to a surface arbitrarily oriented in space:

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_x n_x + \hat{\mathbf{e}}_y n_y + \hat{\mathbf{e}}_z n_z \ , \qquad n_x^2 + n_y^2 + n_z^2 = 1 \ . \qquad (1.17)$$

The net force along this direction is simply

$$\mathbf{t}_{\hat{\mathbf{n}}} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_x)\mathbf{t}_x + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_y)\mathbf{t}_y + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_z)\mathbf{t}_z \qquad = \hat{\mathbf{n}} \cdot \mathbf{s} \ . \qquad (1.18)$$

We can now use the Lagrangian acceleration to write the equivalent of "$F = ma$" or more accurately "$\partial \mathbf{p}/\partial t = \mathbf{F}$", for the fluid element:

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_V \rho \mathbf{u} \mathrm{d}V = \oint_S \mathbf{s} \cdot \hat{\mathbf{n}} \mathrm{d}S \ . \qquad (1.19)$$

We now pull the same tricks as in §1.2.1: use the divergence theorem to turn the surface integral into a volume integral, commute the temporal derivative and volume integral on the RHS, expand carefully the vector operator $\mathbf{u} \cdot \nabla$ acting on $\rho\mathbf{u}$, invoke the arbitrariness of the actual integration volume V, and finally make good use of the continuity equation (1.8), to obtain the differential equation for $\mathbf{u}$:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nabla \cdot \mathbf{s} \ . \qquad (1.20)$$

We now define the **pressure** (units: pascal; 1 Pa$\equiv$ 1 N m^{-2}) as the isotropic part of the force acting perpendicularly on the volume's surfaces, and separate it explicitly from the stress tensor:

$$\mathbf{s} = -p\mathbf{I} + \boldsymbol{\tau} \ , \qquad (1.21)$$

where $\mathbf{I}$ is the identity tensor, and the minus sign arises from the convention that pressures acts on the bounding surface towards the interior of the volume element, and $\boldsymbol{\tau}$ will presently become the **viscous stress tensor**. Since $\nabla \cdot (p\mathbf{I}) = \nabla p$, eq. (1.20) becomes

$$\boxed{\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau}} \ . \qquad (1.22)$$

The next step is to obtained expressions for the components of the tensor $\boldsymbol{\tau}$. The viscous force, which is what $\boldsymbol{\tau}$ stands for, can be viewed as a form of friction acting between contiguous laminae of fluid moving with different velocities, so that we expect it to be proportional to velocity *derivatives*. Consider now the following decomposition of a velocity gradient:

$$\frac{\partial u_k}{\partial x_l} = \underbrace{\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)}_{D_{kl}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k}\right)}_{\Omega_{kl}} \ . \tag{1.23}$$

The first term on the RHS is a **pure shear**, and is described by the (symmetric) **deformation tensor** D_{kl}; the second is a **pure rotation** , and is described by the antisymmetric **vorticity tensor** Ω_{kl}. It can be shown that the latter causes no deformation of the fluid element, *therefore the viscous force can only involve* D_{kl}. A **Newtonian fluid** is one for which the (tensorial) relation between $\boldsymbol{\tau}$ and D_{kl} is linear:

$$\tau_{ij} = f_{ij}(D_{kl}) \ , \qquad i,j,k,l = (1,2,3) \equiv (x,y,z) \tag{1.24}$$

The next step is to invoke the invariance of the physical laws embodied in eq. (1.24) under rotation of the coordinate axes. The mathematics is rather tedious, but at the end of the day you end up with:

$$\tau_{xx} = 2\mu D_{xx} + (\mu_\vartheta - \tfrac{2}{3}\mu)(D_{xx} + D_{yy} + D_{zz}) \tag{1.25}$$
$$\tau_{yy} = 2\mu D_{yy} + (\mu_\vartheta - \tfrac{2}{3}\mu)(D_{xx} + D_{yy} + D_{zz}) \tag{1.26}$$
$$\tau_{zz} = 2\mu D_{zz} + (\mu_\vartheta - \tfrac{2}{3}\mu)(D_{xx} + D_{yy} + D_{zz}) \tag{1.27}$$
$$\tau_{xy} = 2\mu D_{xy} \tag{1.28}$$
$$\tau_{yz} = 2\mu D_{yz} \tag{1.29}$$
$$\tau_{zx} = 2\mu D_{zx} \tag{1.30}$$

where μ and μ_ϑ are the coefficients **dynamical viscosity** and **bulk viscosity**, respectively. Is is often convenient to define a coefficent of **kinematic viscosity** as

$$\nu = \frac{\mu}{\rho} \ , \qquad [\mathrm{m^2\,s^{-1}}] \ . \tag{1.31}$$

In an incompressible flow, the terms multiplying μ_ϑ vanish and it is possible to rewrite the Navier-Stokes equation in the simpler form:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \ . \qquad \text{[incompressible]} \tag{1.32}$$

Note here the presence of a Laplacian operator acting on a **vector** quantity (here **u**); this is only equivalent to the Laplacian acting on the scalar components of **u** in the special case of cartesian coordinates.

Incompressible or not, the behavior of viscous flows will often hinge on the relative importance of the advective and dissipative terms in the Navier-Stokes equation:

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} \qquad \leftrightarrow \qquad \nabla\cdot\boldsymbol{\tau} \ . \tag{1.33}$$

Introducing characteristic length scales u_0, L, ρ_0 and ν_0, dimensional analysis yields:

$$\rho_0\frac{u_0^2}{L} \qquad \leftrightarrow \qquad \frac{1}{L}\rho_0\nu_0\frac{u_0}{L} \ , \tag{1.34}$$

where we made use of the fact that the viscous stress tensor has dimensions $\mu \times D_{ik}$, with $\mu = \rho\nu$ and the deformation tensor D_{ik} has dimension of velocity per unit length (cf. eq. 1.23). The ratio of these two terms is a dimensionless quantity called the **Reynolds Number**:

$$\boxed{\mathrm{Re} = \frac{u_0 L}{\nu_0}} \ . \tag{1.35}$$

This measures the importance of viscous forces versus fluid inertia. It is a key dimensionless parameter in hydrodynamics, as it effectively controls fundamental processes such as the transition to turbulence, as well as more mundane matters such as boundary layer thicknesses.

A few words on boundary conditions; in the presence of viscosity, the flow speed must vanish wherever the fluid is in contact with a rigid surface S:

$$\mathbf{u}(\mathbf{x}) = 0 \ , \qquad \mathbf{x} \in S \ . \tag{1.36}$$

This remains true even in the limit where the viscosity is vanishingly small. For a **free surface** (e.g., the surface of a fluid sphere floating in a vacuum), the normal components of both the flow speed and viscous stress must vanish instead:

$$\mathbf{u}\cdot\hat{\mathbf{n}}(\mathbf{x}) = 0 \ , \qquad \boldsymbol{\tau}\cdot\hat{\mathbf{n}} = 0 \ , \qquad \mathbf{x} \in S \ . \tag{1.37}$$

1.2.4 Angular momentum: the vorticity equation

The "rotation" and "angular momentum" of a fluid system cannot simply be reduced to simple scalars such as angular velocity and moment of inertia, because the application of a torque to a fluid element can alter not just its rotation rate, but also its shape and mass distribution. A more useful measure of "rotation" is the **circulation** Γ about some closed contour γ embedded in and moving with the fluid:

$$\Gamma(t) = \oint_\gamma \mathbf{u}(\mathbf{x},t)\cdot \mathrm{d}\boldsymbol{\ell} = \int_S (\nabla\times\mathbf{u})\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_S \boldsymbol{\omega}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \ , \tag{1.38}$$

where the second equality follows from Stokes' theorem, and the third from the definition of **vorticity**:

$$\boxed{\boldsymbol{\omega} = \nabla\times\mathbf{u}} \ . \tag{1.39}$$

Thinking about flows in terms of vorticity $\boldsymbol{\omega}$ rather than speed $\mathbf{u}$ can be useful because of **Kelvin's theorem**, which states that the circulation Γ along any closed loop γ advected by the moving fluid is a conserved quantity:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0 \ . \tag{1.40}$$

Applying again Stokes' theorem yields the equivalent expression

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\int_S \boldsymbol{\omega}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0} \ , \tag{1.41}$$

stating that the flux of vorticity across any material surface S bounded by γ is also a conserved quantity, both in fact being integral expressions of angular momentum conservation.

An evolution equation for $\boldsymbol{\omega}$ can be obtained via the Navier-Stokes equation, in a particularly illuminating manner in the case of an incompressible fluid ($\nabla\cdot\mathbf{u} = 0$) with constant kinematic viscosity ν, in which case eq. (1.32) can be rewritten as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} == -\nabla\left(\frac{p}{\rho} + \Phi\right) - \nu\nabla\times(\nabla\times\mathbf{u}) \ , \qquad \text{[incompressible]} \tag{1.42}$$

where it was assumed that gravity can be expressed as the gradient of a (gravitational) potential. Taking the curl on each side of this expression then yields:

$$\nabla\times\left(\frac{\partial\mathbf{u}}{\partial t}\right) + \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = \underbrace{\nabla\times\left[\nabla\left(\frac{p}{\rho}+\Phi\right)\right]}_{=0} - \nu\nabla\times\nabla\times(\nabla\times\mathbf{u}) \ , \tag{1.43}$$

then, commuting the time derivative with $\nabla\times$ and making judicious of some vector identities to develop the second term on the LHS, remembering also that $\nabla\cdot\boldsymbol{\omega}=0$, eventually leads to:

$$\boxed{\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t}-\boldsymbol{\omega}\cdot\nabla\mathbf{u}=\nu\nabla^2\boldsymbol{\omega}\ , \qquad \text{[incompressible]}}\ . \tag{1.44}$$

This is the **vorticity equation**, expressing in differential form the conservation of the fluid's angular momentum.

A useful vorticity-related quantity is the **kinetic helicity**, defined as

$$h=\mathbf{u}\cdot\boldsymbol{\omega}\ , \tag{1.45}$$

which measures the amount of twisting in a flow. This will prove an important concept when investigating magnetic field amplification by fluid flows.

1.2.5 Energy: the entropy equation

Omitting to begin with the energy dissipated in heat by viscous friction, the usual accounting of energy flow into and out of a volume element V fixed in space leads to the following differential equation expressing the conservation of the plasma's **internal energy** per unit mass (e, in units J/kg):

$$\frac{\mathrm{D}e}{\mathrm{D}t}+(\gamma-1)e\nabla\cdot\mathbf{u}=\frac{1}{\rho}\nabla\cdot\Big[(\chi+\chi_r)\nabla T\Big]\ , \tag{1.46}$$

where for a perfect gas we have

$$e=\frac{1}{\gamma-1}\frac{p}{\rho}=\frac{1}{\gamma-1}\frac{kT}{\mu m}\ , \tag{1.47}$$

with $\gamma=c_p/c_v$ the ratio of specific heats, and $(\chi+\chi_r)\nabla T$ the heat flux in or out of the fluid element, with χ and χ_r the coefficients of thermal and radiative conductivity, respectively (units: $\mathrm{J\,K^{-1}m^{-1}s^{-1}}$). Equation (1.46) expresses that any variation of the specific energy in a plasma volume moving with the flow (LHS) is due to heat flowing in or out of the volume by conduction or radiation (here in the diffusion approximation). The "extra" term $\propto\nabla\cdot\mathbf{u}$ on the LHS of eq (1.46) embodies the work done against (or by) the pressure force in compressing (or letting expand) the volume element.

It is often convenient to rewrite the energy conservation equation in terms of the plasma's **entropy** $S=\rho^{-\gamma}p$, which allows to rewrite eq. (1.46) in the more compact form:

$$\boxed{\rho T\frac{\mathrm{D}S}{\mathrm{D}t}=\nabla\cdot\Big[(\chi+\chi_r)\nabla T\Big]}\ , \tag{1.48}$$

which states, now unambiguously, that any change in the entropy S as one follows a fluid element (LHS) can only be due to heat flowing out of or into the domain by conduction (RHS). For incompressible fluids eq. (1.48) can be written

$$\rho c_p\frac{\mathrm{D}T}{\mathrm{D}t}=\nabla\cdot\Big[(\chi+\chi_r)\nabla T\Big]\ , \qquad \text{[incompressible]} \tag{1.49}$$

where

$$c_p=T\left(\frac{\partial S}{\partial T}\right)_p\ , \tag{1.50}$$

is the heat capacity at constant pressure.

While this is seldom an important factor in astrophysical flows, in general we must add to the RHS of eq. (1.48) the heat produced by viscous dissipation (and, as we shall see later, by Ohmic dissipation). This is given by the so-called (volumetric) **viscous dissipation function**:

$$\phi_\nu = \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_s}{\partial x_s}\right)^2 + \mu_\vartheta\left(\frac{\partial u_s}{\partial x_s}\right)^2 , \qquad [\mathrm{J\,m^{-3}s^{-1}}] , \tag{1.51}$$

where summation over repeated indices is implied here. Note that since ϕ_ν is positive definite, its presence on the RHS of eq. (1.48) can only increase the fluid element's entropy, which makes perfect sense since friction, which is what viscosity is for fluids, is an irreversible process.

For more on classical hydrodynamics, see the references listed in the bibliography at the end of this chapter.

1.3 The magnetohydrodynamical induction equation

Our task is now to generalize the governing equations of hydrodynamics to include the effects of the electric and magnetic fields, and to obtain evolution equations for these two physical quantities. Keep in mind that electrical charge neutrality, as required by MHD, does not imply that the fluid's microscopic constituents are themselves neutral, but rather that positive and negative electrical charges are present in equal numbers in any fluid element.

The starting point, you guess it I hope, is Maxwell's celebrated equations:

$$\nabla\cdot\mathbf{E} = \frac{\rho_e}{\varepsilon_0} , \qquad [\text{Gauss}'\ \text{Law}] \tag{1.52}$$

$$\nabla\cdot\mathbf{B} = 0 , \qquad [\text{Anonymous}] \tag{1.53}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} , \qquad [\text{Faraday}'\text{s Law}] \tag{1.54}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} , \qquad [\text{Ampere/Maxwell}'\text{s Law}] \tag{1.55}$$

where, in the SI system of units, the electric field is measured in N $\mathrm{C^{-1}}$ ($\equiv$ V $\mathrm{m^{-1}}$), the magnetic field[3] $\mathbf{B}$ in tesla (T). The quantity ρ_e is the electrical charge density (C $\mathrm{m^{-3}}$), and $\mathbf{J}$ is the electrical current density (A $\mathrm{m^{-2}}$). The permittivity ε_0 ($= 8.85\times10^{-12}\mathrm{C^2\,N^{-1}m^{-2}}$ in vacuum) and magnetic permeability μ_0 ($= 4\pi\times10^{-7}$ N $\mathrm{A^{-2}}$ in vacuum) can be considered as constants in what follows, since we will not be dealing with polarisable or ferromagnetic substances.

The first step is (with all due respect to the man) to do away altogether with Maxwell's displacement current in eq. (1.55). This can be justified if the fluid flow is non-relativistic and there are no batteries around being turned on or off, two rather sweeping statement that will be substantiated in §1.5. For the time being we just revert to the original form of Ampère's Law:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} . \tag{1.56}$$

In general, the application of an electrical field $\mathbf{E}$ across an electrically conducting substance will generate an electrical current density $\mathbf{J}$. Ohm's Law postulates that the relationship between $\mathbf{J}$ and $\mathbf{E}$ is linear:

$$\mathbf{J}' = \sigma\mathbf{E}' , \tag{1.57}$$

[3]strictly speaking, $\mathbf{B}$ should be called and the magnetic flux density or somesuch, but on this one we'll stick to common astrophysical usage.

where σ is the electrical conductivity (units: $\mathrm{C^2s^{-1}m^{-3}kg^{-1}} \equiv \Omega^{-1}\mathrm{m}^{-1}$, $\Omega \equiv$Ohm). Here the primes ("$'$") are added to emphasize that Ohm's Law is expected to hold in a conducting substance *at rest*. In the context of a fluid moving with velocity $\mathbf{u}$ (relativistic or not), eq. (1.57) can only be expected to hold in a reference frame comoving with the fluid. So we need to transform eq. (1.57) to the laboratory (rest) frame. In the non-relativitic limit ($u/c \ll 1$, implying $\gamma \to 1$), the usual Lorentz transformation for the electrical current density simplifies to $\mathbf{J}' = \mathbf{J}$, and that for the electric field to $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$, so that Ohm's Law takes on the form

$$\boxed{\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})}\ , \tag{1.58}$$

or, making use of the pre-Maxwellian form of Ampère's Law and reorganizing the terms:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{1}{\mu_0 \sigma}(\nabla \times \mathbf{B})\ . \tag{1.59}$$

We now insert this expression for the electric field into Faraday's Law (1.54) to obtain the very famous **magnetohydrodynamical induction equation**:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})}\ . \tag{1.60}$$

where

$$\eta = \frac{1}{\mu_0 \sigma} \qquad [\mathrm{m^2 s^{-1}}] \tag{1.61}$$

is the **magnetic diffusivity**[4]. The first term on the RHS of eq. (1.60) represents the inductive action of fluid flowing across a magnetic field, while the second term represents dissipation of the electrical currents sustaining the field.

Keep in mind that any solution of eq. (1.60) must also satisfy eq. (1.53) at all times. It can be easily shown (try it!) that if $\nabla \cdot \mathbf{B} = 0$ at some initial time, the form of eq. (1.60) guarantees that zero divergence will be maintained at all subsequent times[5]

1.4 Scaling analysis

The evolution of a magnetic field under the action of a prescribed flow $\mathbf{u}$ will depend greatly on whether or not the inductive term on the RHS of eq. (1.60) dominates the diffusive term. Under what conditions will this be the case? We seek a first (tentative) answer to this question by performing a dimensional analysis of eq. (1.60); this involves replacing the temporal derivative by $1/\tau$ and the spatial derivatives by $1/\ell$, where τ and ℓ are time and length scales that suitably characterizes the variations of *both* $\mathbf{u}$ *and* $\mathbf{B}$:

$$\frac{\mathbf{B}}{\tau} = \frac{u_0 \mathbf{B}}{\ell} + \frac{\eta \mathbf{B}}{\ell^2}\ , \tag{1.62}$$

where B and u_0 are a "typical" values for the flow velocity and magnetic field strength over the domain of interest. The ratio of the first to second term on the RHS of eq. (1.62) is a dimensionless quantity known as the **magnetic Reynolds number**[6]:

$$\boxed{\mathrm{R}_m = \frac{u_0 \ell}{\eta}}\ , \tag{1.63}$$

[4] A note of warning: some MHD textbooks, included the Goedbloed & Poedts tome cited in the bibliography, use the symbol "η" for the inverse conductivity (units $\Omega\,$m), so that the dissipative term on the RHS of the induction equation retains a μ_0^{-1} prefactor. The Davidson book uses the same definition as here... but the symbol λ! Be careful.

[5] This is true under exact arithmetic; if numerical solutions to eq. (1.60) are sought, care must be taken to ensure $\nabla \cdot \mathbf{B} = 0$ as the solution is advanced in time.

[6] Not the structural similarity with the usual viscous Reynolds number defined in §1.2.3, with the magnetic diffusivity η replacing the kinematic viscosity ν in the denominator. Had we not absorbed μ_0 in our definition of η, the magnetic permeability μ_0 would appear in the numerator of the magnetic Reynolds number, which I personally find objectionable.

which measures the relative importance of induction versus dissipation *over length scales of order* ℓ. Note that R_m does not depend on the magnetic field strength, a direct consequence of the linearity (in $\mathbf{B}$) of the MHD induction equation. Our scaling analysis simply says that in the limit $\mathrm{R}_m \gg 1$, induction by the flow dominates the evolution of $\mathbf{B}$, while in the opposite limit of $\mathrm{R}_m \ll 1$, induction makes a negligible contribution and $\mathbf{B}$ simply decays away under the influence of Ohmic dissipation.

One may anticipate great simplifications of magnetohydrodynamics if we operate in either of these limits. If $\mathrm{R}_m \ll 1$, only the second term is retained on the RHS of eq. (1.62), which leads immediately to

$$\boxed{\tau = \frac{\ell^2}{\eta}} \ , \tag{1.64}$$

a quantity known as the **magnetic diffusion time**. It measures the time taken for a magnetic field contained in a volume of typical linear dimension ℓ to dissipate and/or diffusively leak out of the volume. Now, for most astrophysical objects, this timescale turns out to be quite large, indeed often larger than the age of the universe! (see Table 1.2). This is not so much because astrophysical plasmas are such incredibly good electrical conductors, but rather because astrophysical objects tend to be very, very large.

The opposite limit $\mathrm{R}_m \gg 1$, defines the **ideal MHD** limit. Then it is the first term that is retained on the RHS of eq. (1.62), so that

$$\tau = \ell/u_0 \ , \tag{1.65}$$

corresponding to the **turnover time** associated with the flow $\mathbf{u}$. Note already that under ideal MHD, the only non-trivial (i.e., $\mathbf{u} \neq 0$ and $\mathbf{B} \neq 0$) steady-state ($\partial/\partial t = 0$) solutions of the MHD equation are only possible for field-aligned flows.

Table 1.2 below lists estimates of the magnetic Reynolds number (and related physical quantities) for the various astrophysical systems considered earlier in Table 1.1[7]:

Table 1.2
Properties of some astrophysical objects and flows

System/flow	L [km]	σ [Ω^{-1}m^{-1}]	η [m^{2}s^{-1}]	τ [yr]	u [km/s]	R_m
Solar interior	10^6	10^4	100	10^9	0.1	10^9
Solar atmosphere	10^3	10^3	1000	10^2	1	10^6
Solar corona	10^5	10^6	1	10^8	10	10^{12}
Solar wind (1 AU)	10^5	10^4	100	10^8	300	10^{11}
Molecular cloud	10^{14}	10^2	10^4	10^{17}	100	10^{18}
Interstellar medium	10^{16}	10^3	1000	10^{22}	100	10^{21}
Sphere of copper	10^{-3}	10^8	10^{-1}	10^{-7}	—	—

The magnetic Reynolds number is clearly huge in all cases, which would suggest that the ideal MHD limit is the one most applicable to all these astrophysical systems. But things are not so simple. From a purely mathematical point of view, taking the limit $\mathrm{R}_m \to \infty$ of the MHD induction equation is problematic, because the order of the highest spatial derivatives decreases by one. This situation is similar to the behavior of viscous flows at very high Reynolds number:

[7]Choices for length scale ℓ ($\equiv L$) as in Table 1.1. Velocity estimates correspond to large convective cells (solar interior), granulation (photosphere), solar wind speed (corona and solar wind), and turbulence (molecular clouds and interstellar medium). All these numbers (especially the turbulent velocity estimates) are again very rough, and rounded to the nearest factor of ten. The magnetic diffusivity estimates given for molecular clouds and interstellar medium depend critically on the assumed degree of ionization, and so are also very rough.

solutions to eq. (1.60) with $\eta \to 0$ in general **do not** smoothly tend towards solutions obtained for $\eta = 0$. Moreover, the distinction between the two physical regimes $\mathrm{R}_m \ll 1$ and $\mathrm{R}_m \gg 1$ is meaningful as long as one can define a suitable R_m for the flow as a whole, which, in turn, requires one to estimate, *a priori*, a length scale ℓ that adequately characterizes the evolving magnetic field at all time and throughout the spatial domain of interest. As we proceed it will become clear that this is not always straightforward, or even possible. Finally, the scaling analysis does away entirely with the geometrical aspects of the problem, by substituting $u_0 B$ for $\mathbf{u} \times \mathbf{B}$; yet there are situations (e.g. a field-aligned flow) where even a very large $\mathbf{u}$ has no inductive effect whatsoever, in which case the induction equation assumes the mathematical form

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\eta \nabla \times \mathbf{B}) \ , \tag{1.66}$$

even though R_m may be very large, and $\mathbf{B}$ evolves on the (long) magnetic diffusion timescale (1.64) rather than on the (short) turnover time.

1.5 The Lorentz force

Getting to eq. (1.60) was pretty easy (because we summarily swept the displacement current under the rug), but it represents only half (in fact the easy half) of our task; we must now investigate the effect of the magnetic field on the flow $\mathbf{u}$; and this, it turns out, is the tricky part of the MHD approximation.

You will certainly recall that the **Lorentz force** acting on an electrically charge particle moving at velocity $\mathbf{u}$ in a region of space permeated by electric and magnetic fields is given by

$$\mathbf{f} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \ , \qquad [\mathrm{N}] \ . \tag{1.67}$$

where q is the electrical charge. Consider now a volume element ΔV containing many such particles; in the continuum limit, the total force per unit volume ($\mathbf{F}$) acting on the volume element will be the sum of the forces acting on each individual charged constituents divided by the volume element:

$$\begin{aligned} \mathbf{F} = \frac{1}{\Delta V}\sum_k \mathbf{f}_k &= \frac{1}{\Delta V}\sum_k q_k(\mathbf{E} + \mathbf{u_k} \times \mathbf{B}) \\ &= \left(\frac{1}{\Delta V}\sum_k q_k\right)\mathbf{E} + \left(\frac{1}{\Delta V}\sum_k q_k \mathbf{u_k}\right) \times \mathbf{B} \\ &= \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B} \ , \qquad [\mathrm{N\,m^{-3}}] \ . \end{aligned} \tag{1.68}$$

where the last equality follows from the usual definition of charge density and electrical current density. At this point you might be tempted to eliminate the term proportional to $\mathbf{E}$, on the grounds that in MHD we are dealing with a globally neutral plasma, meaning $\rho_e = 0$, therefore $\rho_e \mathbf{E} \equiv 0$ and that's the end of it. That would be way too easy...

Let's begin by taking the divergence on both side of the generalized form of Ohm's Law (eq. (1.58)). We then make use of Gauss's Law (eq. (1.53)) to get rid of the $\nabla \cdot \mathbf{E}$ term, and of the charge conservation Law

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{1.69}$$

to get rid of the $\nabla \cdot \mathbf{J}$ term. The end result of all this physico-algebraical juggling is the following expression:

$$\frac{\partial \rho_e}{\partial t} + \frac{\rho_e}{(\varepsilon_0/\sigma)} + \sigma \nabla \cdot (\mathbf{u} \times \mathbf{B}) = 0 \ . \tag{1.70}$$

The combination ε_0/σ has units of time, and is called the **charge relaxation time**, henceforth denoted τ_e. It is the timescale on which charge separation takes place in a conductor if an electric field is suddenly turned on. For most conductors, this a very small number, of order 10^{-18} s !! This is because the electrical field reacts to the motion of electric charges at the speed of light (in the substance under consideration, which is slower than in a vacuum but still mighty fast). Indeed, in a conducting fluid at rest ($\mathbf{u} = 0$) the above expression integrates readily to

$$\rho_e(t) = \rho_e(0) \exp(-t/\tau_e) \ , \tag{1.71}$$

thus the name "relaxation time" for τ_e.

Now let us consider the case of a slowly moving fluid, in the sense that it is moving on a timescale much larger than τ_e; this means that the induced electrical field will vary on a similar timescale (at best), and therefore the time derivative of ρ_e can be neglected in comparison to the ρ_e/τ_e term in eq. (1.70), leading to

$$\rho_e = \varepsilon_0 \nabla \cdot (\mathbf{u} \times \mathbf{B}) \ . \tag{1.72}$$

This indicates that a finite charge density can be sustained inside a *moving* conducting fluid. The associated electrostatic force per unit volume, $\rho_e\mathbf{E}$, is definitely non-zero but turns out to much *smaller* than the magnetic force. Indeed, a dimensional analysis of eq. (1.68), using eq. (1.72) to estimate ρ_e, gives:

$$\rho_e\mathbf{E} \sim \left(\frac{\varepsilon_0 uB}{\ell}\right)\left(\frac{J}{\sigma}\right) \sim \left(\frac{u\tau_e}{\ell}\right) JB \ , \tag{1.73}$$

$$\mathbf{J} \times \mathbf{B} \sim JB \ , \tag{1.74}$$

where Ohm's Law was used to express $\mathbf{E}$ in terms of $\mathbf{J}$, and once again ℓ is a typical length scale characterizing the variations of the flow and magnetic field. The ratio of electrostatic to magnetic force is thus of order $u\tau_e/\ell$. Now $\tau_e \ll 1$ to start with, and for non-relativistic fluid motion we can expect that the flow's turnover time ℓ/u is much larger than the crossing time for an electromagnetic disturbance $\sim \ell/c \sim \tau_e$; both effects conspire to render the electrostatic force absolutely minuscule compared to the magnetic force, so that eq. (1.68) becomes

$$\boxed{\mathbf{F} = \mathbf{J} \times \mathbf{B} \ , \qquad \text{[MHD approximation]}} \ . \tag{1.75}$$

and this must be added to the RHS of the Navier-Stokes equation (1.22)... with a $1/\rho$ prefactor so we get a force per unit mass, rather than per unit volume.

Now, getting back to this business of having dropped the displacement current in the full Maxwellian form of Ampère's Law (eq. (1.55)); it can now be all justified on the grounds that the time derivative of the charge density can be neglected in the non-relativistic limit. Indeed, to be consistent the charge conservation equation (1.69) now reduces to

$$\nabla \cdot \mathbf{J} = 0 \ ; \tag{1.76}$$

taking the divergence on both sides of eq. (1.55) then leads to

$$\nabla \cdot \mathbf{J} = -\varepsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t}\right) = \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \frac{\partial \rho_e}{\partial t} \ ; \tag{1.77}$$

this demonstrates that dropping the time derivative of the charge density is equivalent to neglecting Maxwell's displacement current in eq. (1.55). To sum up, provided we exclude very rapid transient events (such as turning a battery on or off, or any such process which would generate a large $\partial\rho_e/\partial t$), under the MHD approximation the following statements all hold true:

- The fluid motions are non-relativistic;
- The electrostatic force can be neglected as compared to the magnetic force;
- Maxwell's displacement current can be neglected.

1.6 Joule heating

In the presence of finite electrical conductivity, the volumetric heating associated with the dissipation of electric currents must be included on the RHS of the energy equation, in the form of the so-called **Joule heating function**:

$$\phi_B = \frac{\eta}{\mu_0}(\nabla \times \mathbf{B})^2 \ , \qquad [\mathrm{J\,m^{-3}s^{-1}}] \ . \tag{1.78}$$

Note however that in very nearly all astrophysical circumstances, Joule heating makes an insignificant contribution to the energy budget. When it occurs, heating by magnetic energy dissipation, such as in flares, involves dynamical mechanisms that lead to effective dissipation far more rapid and efficient than Joule heating.

1.7 The full set of MHD equations

For the record, we now collect the set of partial differential equations governing the behavior of magnetized fluids in the MHD limit:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \ , \tag{1.79}$$

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \frac{1}{\mu_0\rho}(\nabla\times\mathbf{B})\times\mathbf{B} + \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau} \ , \tag{1.80}$$

$$\frac{\mathrm{D}e}{\mathrm{D}t} + (\gamma-1)e\nabla\cdot\mathbf{u} = \frac{1}{\rho}\Big[\nabla\cdot\Big((\chi+\chi_r)\nabla T\Big) + \phi_\nu + \phi_B\Big] \ , \tag{1.81}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B} - \eta\nabla\times\mathbf{B}) \ . \tag{1.82}$$

Equations (1.79)—(1.82) are further complemented by the two constraint equations:

$$\nabla\cdot\mathbf{B} = 0 \ , \tag{1.83}$$

$$p = f(\rho, T, ...) \ , \tag{1.84}$$

and suitable expressions for the viscous stress tensor and for the physical coefficient ν, χ, η, etc. Note that gravity $\mathbf{g}$ is explicitly included on the RHS of (1.80), that e is the specific energy of the plasma (magnetic energy will be dealt with separately shortly), and that eq. (1.84) is just some generic form for an equation of state linking the pressure to the properties of the plasma such as density, temperature, chemical composition, etc.

This is it in principle, but in what follows we shall seldom solve these equations in this complete form. In the parameter regime characterizing most astrophysical fluids, we usually have $\mathrm{Re} \gg 1$, which means that the $(\mathbf{u}\cdot\nabla\mathbf{u})$ term in eq. (1.80) will play important role; this, in turn, means turbulence, already in itself an unsolved problem even for unmagnetized fluids. There is also a strong nonlinear coupling between eqs. (1.80) and eqs. (1.82), so that the turbulent cascade involves both the flow and magnetic field. Finally, with both $\mathrm{Re} \gg 1$ and $\mathrm{R}_m \gg 1$, astrophysical flows will in general develop structures on length scales very much smaller than that characterizing the system under study, so that even fully numerical solutions of the above set of MHD equations will tax the power of the largest extant massively parallel computers, and will continue to do so in the foreseeable future; which is why judicious geometrical and/or physical simplification remains a key issue in the art of astrophysical magnetohydrodynamics... and will also continue to remain so in the same foreseeable future!

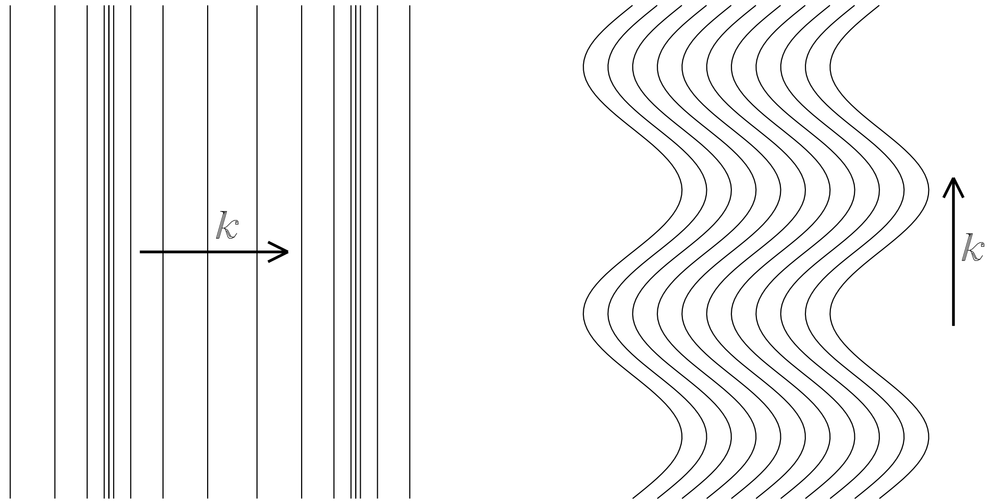

Figure 1.4: The two fundamental MHD wave modes in a uniform background magnetic field: (A) magnetosonic mode, and (B) Alfvén mode. The wave vector **k** is indicated as a thick arrow, and highlights the fact that the magnetosonic mode is a longitudinal wave, while the Alfvén mode is a transverse wave. In the presence of plasma, the magnetosonic mode breaks into two submodes, according to the phasing between the magnetic pressure and gas pressure perturbations (see text).

1.8 MHD waves

Although it looks innocuous enough, the magnetic force in the MHD approximation has some rather complex consequences for fluid flows, as we will have ample occasions to verify throughout this course. One particularly intricate aspects relates to the types of **waves** that can be supported in a magnetized fluid; in a classical unmagnetized fluid, one deals primarily with sound waves (pressure acting as a restoring force), gravity waves (gravity acting as restoring force), or Rossby waves (Coriolis as a restoring force). It turns out that the Lorentz force introduces not one, but really two additional restoring forces.

Making judicious use of eqs. (1.53) and (1.56), together with some classical vector identities, eq. (1.75) can be rewritten as

$$\mathbf{F} = \frac{1}{\mu_0}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla(\mathbf{B}^2)\right] , \tag{1.85}$$

where $\mathbf{B^2} \equiv \mathbf{B}\cdot\mathbf{B}$. The first term on the RHS is the **magnetic pressure**, and the second the **magnetic tension**. The general idea is illustrated on Figure 1.4. Fluctuations in magnetic pressure can propagate as a longitudinal wave, much as a sound wave, as depicted on Fig. 1.4A. In fact, two such **magnetosonic** waves modes actually exist, according to whether the magnetic pressure fluctuation is in phase with the gas pressure fluctuation (the so-called fast mode), or in antiphase (the slow mode). In addition, magnetic tension can produce a restoring force that allows the propagation of wave-on-a-string-like transverse waves, known as **Alfvén waves**, as illustrated on Fig. 1.4B.

Problem 1.7 gets you to calculate some basic characteristics of Alfvén waves propagating in a homogeneous medium threaded by a uniform magnetic field. If all goes well, you should find that small-amplitude Alfvén waves travel with a speed

$$u_A = \frac{B_0}{\sqrt{\mu_0\rho}} , \tag{1.86}$$

where B_0 is the magnitude of the (uniform) magnetic field along which the wave is propagating, and ρ is the (constant) fluid density. We will not be dealing much with magnetosonic waves in this course, but we will return to Alfvén waves in part II, when we examine their dynamical impact on the acceleration of wind-like outflows from the sun and stars.

1.9 Magnetic energy

Consider the expression resulting from dotting **B** into the induction equation (1.60), integrating over the spatial domain (V) under consideration, and making judicious use of various well-known vector identities and of Gauss' theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{\mathbf{B}^2}{2\mu_0}\mathrm{d}V = -\int_S (\mathbf{S}\cdot\hat{\mathbf{n}})\,\mathrm{d}S - \int_V (\mathbf{u}\cdot\mathbf{F})\,\mathrm{d}V - \int_V \sigma^{-1}\mathbf{J}^2\,\mathrm{d}V \ , \tag{1.87}$$

where **E** is the electric field, and $\hat{\mathbf{n}}$ is a outward-directed unit vector normal to the boundary surface S. The vector quantities **S**, **L** and **J** are the Poynting flux, Lorentz force and current density, respectively. Recall that in the MHD limit these take the form:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \ , \tag{1.88}$$

$$\mathbf{F} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} \ , \tag{1.89}$$

$$\mu_0\mathbf{J} = \nabla\times\mathbf{B} \ . \tag{1.90}$$

We also made use of the fact that in the MHD approximation, the net current **J** is expressed via the generalized form of Ohm's law as the sum of the conduction and induction currents:

$$\mathbf{J} = \sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \ . \tag{1.91}$$

Examine now the three terms on the RHS of eq. (1.87); the first is the Poynting flux component into the domain, integrated over the domain boundaries, i.e., the flux of electromagnetic energy in (integrand < 0) or out (integrand > 0) of the domain. This term evidently vanishes in the absence of applied magnetic or electric fields on the boundaries. The second is the work done by the Lorentz force (**F**) on the flow. In general this term can be either positive or negative; in the context of magnetic driving of stellar winds (part II of the course) we are concerned with the $\mathbf{u}\cdot\mathbf{F} > 0$ situation[8], while in the dynamo context (part III of the course) we are interested in the $\mathbf{u}\cdot\mathbf{F} < 0$ situation, where the flow transfers energy to the magnetic field. The third term is evidently always negative, and represents the rate of energy loss due to Ohmic dissipation. Equations (1.87) then naturally leads to interpret the quantity $\mathbf{B}^2/2\mu_0$ as the magnetic energy density, since the LHS of eq. (1.87) is clearly the rate of change of the total **magnetic energy** ($\mathcal{E}_\mathrm{B}$) within the domain:

$$\boxed{\mathcal{E}_\mathrm{B} = \frac{1}{2\mu_0}\int_V \mathbf{B}^2\mathrm{d}V} \ . \tag{1.92}$$

1.10 Magnetic flux freezing and Alfvén's theorem

Let us return to the differential form of Faraday's Law:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \ . \tag{1.93}$$

[8] which is also quite relevant to the design of magnetic pumps for electrically conducting fluids. This has received quite a bit of attention in light of the use of liquid sodium to cool the core of nuclear reactors.

Project now each side of this expression onto a unit vector normal to some surface S fixed in space and bounded by a closed countour γ, integrate over S, and apply Stokes' theorem to the LHS:

$$\int_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} \mathrm{d}S = \oint_\gamma \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\int_S \left(\frac{\partial \mathbf{B}}{\partial t} \right) \cdot \hat{\mathbf{n}} \mathrm{d}S \ . \tag{1.94}$$

So far the surface S remains completely arbitrary. If it is fixed in space, then we get the usual integral form of Faraday's Law:

$$\oint_\gamma \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \hat{\mathbf{n}} \mathrm{d}S \ , \tag{1.95}$$

with the LHS corresponding to the electromotive force, and the RHS to the time variation of the magnetic flux (Φ_B). If we now assume instead that the surface S is a material surface moving with the fluid, then (1) we must substitute the Lagrangian operator $\mathrm{D}/\mathrm{D}t$ for the partial derivative on the RHS of eq. (1.95); and (2) we are allowed to invoke Ohm's Law to substitute $\mathbf{J}$ for $\mathbf{E}$ on the RHS since any point of the (material) contour is by definition co-moving with the fluid:

$$\frac{1}{\sigma} \oint_\gamma \mathbf{J} \cdot \mathrm{d}\ell = -\frac{\mathrm{D}}{\mathrm{D}t} \int_S \mathbf{B} \cdot \hat{\mathbf{n}} \mathrm{d}S \ . \tag{1.96}$$

Now, obviously, in the limit of infinite conductivity we have

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_S \mathbf{B} \cdot \hat{\mathbf{n}} \mathrm{d}S = 0 \ . \tag{1.97}$$

This states that in the ideal MHD limit $\sigma \to \infty$, the magnetic flux threading any (open) surface is a conserved quantity as the surface is advected (and possibly deformed) by the flow. This results is known as **Alfvén's theorem**. Note in particular that in the limit of an infinitisemal surface pierced by "only one" fieldline, Alfvén's theorem is equivalent to saying that magnetic fieldline must move in the same way as fluid elements; it is customary to stay that the magnetic field is "frozen" into the fluid. In this manner it behaves just like vorticity in the inviscid limit $\nu \to 0$. And like in the case of vorticity, sheared flows can amplify magnetic fields by stretching, a subject we will investigate in all great details in Part III of these class notes.

1.11 Magnetic helicity

In anology with fluid helicity, one can define the **magnetic helicity** as

$$h_B = \mathbf{A} \cdot \mathbf{B} \ . \tag{1.98}$$

Note that the magnetic vector potential $\mathbf{A}$ is playing here the role of the flow field $\mathbf{u}$ in eq. (1.45). A substantial amount of vector algebra can show that the total magnetic helicity in any volume V of magnetized fluid

$$\mathcal{H}_B = \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V \ , \tag{1.99}$$

is a conserved quantity under ideal MHD ($\sigma \to \infty$). In direct analogy to fluid helicity, this is a measure of twist of magnetic fieldlines, and/or of topological linkage between distinct flux systems. A closely related quantity is the **current helicity**:

$$\mathcal{H}_J = \int_V \mathbf{J} \cdot \mathbf{B} \, \mathrm{d}V \ , \tag{1.100}$$

which, in the astrophysical context, is often an easier quantity to determine observationally. Note that since $\mathbf{J} \propto \nabla \times \mathbf{B}$ and $\mathbf{B} \propto \nabla \times \mathbf{A}$, a proportionality between $\mathcal{H}_B$ and $\mathcal{H}_J$ is expected, and the two quantities should certainly have the same sign for a given physical system.

1.12 Mathematical representations of magnetic fields

We close this heavy-duty chapter with a somewhat disorganized collection of mathematical and physical properties of the vector magnetic field, which will be of great use in chapters to follow.

1.12.1 Pseudo-vectors and solenoidal vectors

It is worth distinguishing between real vectors (also called axial vectors) and **pseudo-vectors**, the latter class including the magnetic field vector. Real vectors remain invariant upon inversion of the (3D) coordinates about the origin, i.e., $\mathbf{x} \to -\mathbf{x}$, hereafter thinking in cartesian coordinates to ease the discussion. This will leave the "physical" direction in space of a true vector (like a velocity $\mathbf{u}$) unchanged, since both the coordinate unit vectors and the components of the velocity will change sign:

$$\mathbf{u}' = (-u_x)(-\hat{\mathbf{e}}_x) + (-u_y)(-\hat{\mathbf{e}}_y) + (-u_z)(-\hat{\mathbf{e}}_z) = \mathbf{u} \ . \tag{1.101}$$

However, in terms of vector products, curl operators, orientation of surfaces and so on, the coordinate inversion will take us from a right-handed coordinate system to a left-handed system. This implies that a vector like the magnetic field must remain invariant under coordinate inversion. This can be appreciated by considering the expression for the magnetic force acting on a charge q moving at velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$:

$$\mathbf{f} = q\,\mathbf{u} \times \mathbf{B} \ ; \tag{1.102}$$

we just argued that the components of $\mathbf{f}$ and $\mathbf{u}$ will change sign under coordinate inversion; therefore the magnetic field components must *not* change sign under coordinate inversion, for eq. (1.102) to remain valid (physical laws do not care about our coordinate conventions!). One must conclude that upon coordinate inversion, the direction of a vector field such as $\mathbf{B}$ immediately flips! So the Earth's north magnetic pole instantly becomes the south magnetic pole[9]. Weird behavior for a vector, which is why such vectors inherit the prefix "pseudo".

Pseudo or not, there are numerous vectors fields of physical interest out there that have the property that their divergence vanishes; the magnetic field is evidently such a vector field, as per our second Maxwell equation (1.53). The fluid vorticity (§1.2.4) is clearly another. Any vector field ($\mathbf{G}$, say) satisfying $\nabla \cdot \mathbf{G} = 0$ is called a **solenoidal vector**.

Soleinodal vectors have a very interesting property related to the conservation of their flux across material surfaces transported and deformed by a flow field $\mathbf{u}$. They can be shown to satisfy the following **kinematic theorem**:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{S_m} \mathbf{G}\cdot\hat{\mathbf{n}}\mathrm{d}S = \int_{S_m}\left[\frac{\partial \mathbf{G}}{\partial t} - \nabla\times(\mathbf{u}\times\mathbf{G})\right]\cdot\hat{\mathbf{n}}\mathrm{d}S \ . \tag{1.103}$$

This is simply saying that the net variation of the flux (LHS) can be due either to intrinsic time-variation of the vector field (first term in the square brackets on the RHS) or to deformation of the material surface S_m by the flow $\mathbf{u}$ (second term).

Note that we could have arrived at Alfvén's theorem (§1.10) starting from this kinematic theorem for solenoidal vector fields, as applied to $\mathbf{B}$:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{S_m} \mathbf{B}\cdot\hat{\mathbf{n}}\mathrm{d}S = \int_{S_m}\left[\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathbf{B})\right]\cdot\hat{\mathbf{n}}\mathrm{d}S \tag{1.104}$$

Obviously, the quantity within square brackets on the RHS will vanish as per our MHD induction equation (1.60) written in the ideal limit $\eta \to 0$, which gets us directly to eq. (1.97). You will recall, of course, that Ohm's Law is indeed already embodied in the MHD induction equation, so this is really getting to the same result by two mathematically distinct but physically equivalent paths.

[9] Does this mean that your compass needle will instantly rotate by 180 degrees? Think about that one a bit...

1.12.2 The vector potential

It will often prove useful to work with the MHD induction equation written in terms of a vector potential $\mathbf{A}$ (units T m) such that $\mathbf{B} = \nabla \times \mathbf{A}$. Equation (1.60) is then readily integrated to

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times (\nabla \times \mathbf{A}) - \eta \nabla \times (\nabla \times \mathbf{A}) + \nabla \Phi \ , \tag{1.105}$$

where, in "uncurling" the induction equation we may elect to append the gradient of a scalar function to the RHS, with no effect on $\mathbf{B}$. This additional term may contribute to the electric field $\mathbf{E}$, however, and so Φ is conveniently regarded as the electrostatic potential[10]. Clearly, any solution of eq. (1.105) identically satisfies the solenoidal constraint $\nabla \cdot \mathbf{B} = 0$.

1.12.3 Axisymmetric magnetic fields

In many astrophysical situations to be encountered in subsequent chapters we will facing astrophysical magnetofluid systems that show symmetry about an axis, in fact usually a rotational axis. For example, the sun's differential rotation and meridional circulation, as inferred from surface measurements and helioseismology, are very closely axisymmetric on the largest spatial scales. In spherical polar coordinates (r, θ, ϕ), the most general axisymmetric ($\partial/\partial\phi = 0$) magnetic field and flow can be written as

$$\mathbf{u}(r, \theta, t) = \frac{1}{\rho} \nabla \times (\Psi(r, \theta, t) \hat{\mathbf{e}}_\phi) + \varpi \Omega(r, \theta, t) \hat{\mathbf{e}}_\phi \tag{1.106}$$

$$\mathbf{B}(r, \theta, t) = \nabla \times (A(r, \theta, t) \hat{\mathbf{e}}_\phi) + B(r, \theta, t) \hat{\mathbf{e}}_\phi \tag{1.107}$$

where $\varpi = r \sin\theta$. Here the vector potential component A and stream function Ψ define the **poloidal** components of the field and flow, i.e., the component contained in meridional (r, θ) planes. The azimuthal component B is often called the **toroidal field**, and Ω is the angular velocity (units rad s^{-1}). Evidently eqs. (1.106)—(1.107) satisfies the constraints $\nabla \cdot (\rho \mathbf{u}) = 0$ (mass conservation in a steady flow) and $\nabla \cdot \mathbf{B} = 0$ by construction.

A practical advantage of this so-called mixed representation is that it allows the separation of the (vector) MHD induction equation into *two* components for the 2D scalar fields A and B:

$$\frac{\partial}{\partial t}(\varpi A) + \mathbf{u}_p \cdot \nabla(\varpi A) = \varpi \eta \left(\nabla^2 - \frac{1}{\varpi^2} \right) A \ , \tag{1.108}$$

$$\frac{\partial}{\partial t}\left(\frac{B}{\varpi}\right) + \mathbf{u}_p \cdot \nabla \left(\frac{B}{\varpi}\right) = \frac{\eta}{\varpi} \left(\nabla^2 - \frac{1}{\varpi^2} \right) B + \frac{1}{\varpi} (\nabla \eta) \times (B \hat{\mathbf{e}}_\phi)$$

$$- \left(\frac{B}{\varpi}\right) \nabla \cdot \mathbf{u}_p + \mathbf{B}_p \cdot \nabla \Omega \ , \tag{1.109}$$

where $\mathbf{B}_p$ and $\mathbf{u}_p$ are notational shortcuts for the poloidal field and meridional flow. Notice that the vector potential A evolves in a manner entirely independent of the toroidal field B, the latter being conspicuously absent on the RHS of eq. (1.108). This is not true of the toroidal field B, which is well aware of the poloidal field's presence via the $\nabla\Omega$ shearing term.

On numerous occasions in this and subsequent chapters we will seek solutions to eqs. (1.108)—(1.109) inside a sphere (radius R) of magnetized fluid; in the "exterior" $r > R$ there is only vacuum, which implies vanishing electric currents. In practice we will need to match whatever solution we compute in $r < R$ to a current-free solution in $r > R$; such a solution must satisfy

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = 0 \ . \tag{1.110}$$

[10] In most (but not all!) situations dealt with in the following pages, Φ can (and will) be set to zero without objectionable consequences.

For an axisymmetric system eq. (1.110) then translates into

$$\left(\nabla^2 - \frac{1}{\varpi^2}\right) A(r,\theta,t) = 0 \ , \tag{1.111}$$

$$B(r,\theta,t) = 0 \ . \tag{1.112}$$

Solutions to eq. (1.111) have the general form

$$A(r,\theta,t) = \sum_{l=1}^{\infty} a_l \left(\frac{R}{r}\right)^{l+1} P_l^1(\cos\theta) \qquad r > R \ , \tag{1.113}$$

where the P_l^1 are the associated Legendre functions of order 1 and l is a positive integer. Solutions to eqs. (1.108)–(1.109) computed within the sphere must then be smoothly matched to eqs. (1.112)—(1.113) in the exterior. In particular, the vector potential A must be continuous up to its first derivative normal to the surface, so that the magnetic field component tangential to the surface remains continuous across $r = R$. Regularity of the magnetic field on the symmetry axis ($\theta = 0$) requires that we set $B = 0$ there. Without any loss of generality, we can also set $A = 0$ on the axis.

1.12.4 Force-free magnetic fields

In many astrophysical systems, the magnetic field dominates the dynamics and energetics of the system. Left to itself, such a system would tend to evolve to a force-free state described by

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} = 0 \ . \tag{1.114}$$

Broadly speaking, this can be achieved in two physically distinct ways (excluding the trivial solution $\mathbf{B} = 0$). The first is $\mathbf{J} = 0$ throughout the system. Then Ampère's Law becomes $\nabla \times \mathbf{B} = 0$, which means that, as with the electric field in electrostatic, $\mathbf{B}$ can be expressed as the gradient of a potential. Such a magnetic field is called a **potential field**. Substitution into Gauss' Law then yields a Laplace-type problem:

$$\mathbf{B} = \nabla\Phi \ , \qquad \nabla^2\Phi = 0 \ , \qquad [\text{Potential field}] \ , \tag{1.115}$$

with Ampère's Law being trivially satisfied (0 = 0!). Alternately, a system including a non-zero current density can still be force free, provided the currents flow everywhere parallel to the magnetic field, i.e.,

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \ , \tag{1.116}$$

where α need not necessarily be a constant, i.e., it can vary from one fieldline to another, vary in space, and even depend on the (local) value of $\mathbf{B}$. Imagine now a situation where, in some domain (for example, the exterior of a star), we are provided with a boundary condition on $\mathbf{B}$ and the task is to construct a force-free field. Adopting the potential field *anzatz* can lead to very different reconstructions that if we adopt instead eq. (1.116), given that in the latter case one is free to specify any electric current distribution within the domain, as long as $\mathbf{J}$ remains parallel to $\mathbf{B}$.

A very important result in this context is known as **Aly's Theorem**; it states that in a semi-infinite domain with $\mathbf{B}_\perp$ imposed at the boundary and $\mathbf{B} \to 0$ as $\mathbf{x} \to \infty$, the (unique) potential field solution satisfying the boundary conditions has a magnetic energy that is *lower* than any of the (multiple) solutions of eq. (1.116) that satisfy the same boundary conditions, even with complete freedom to specify $\alpha(\mathbf{x})$ within the domain. This poses a strict limit to the amount of magnetic energy stored into a system that can actually be tapped into to power astrophysically interesting phenomena.

Problems:

1. Obtain the charge conservation equation (1.69) by following the general logic used in §1.2.1 to obtain the continuity equation (1.8).
2. Fill in the missing mathematical steps leading to eq. (1.70)
3. Fill in the missing mathematical steps leading to eq. (1.85)
4. Obtain equations (1.108) and (1.109) by substitution of eqs. (1.106) and (1.107) into the MHD induction equation (1.60). Hint: the induction equation is a vector equation; terms "oriented" in the ϕ-direction must cancel one another independently of terms oriented perpendicular to the ϕ-direction.
5. This one is a challenge to your vector algebraic skills. The idea is to mathematically prove the statement made in §1.11, namely that the magnetic helicity in a volume of plasma is a conserved quantity in the idea MHD limit ($\sigma \to \infty$). Remembering that the volume itself is carried (and possibly deformed) by the flow, you must show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V = 0 \ . \tag{1.117}$$

   If you get stuck, take a look at the Goedbloed & Poedts textbook cited below for helpful hints...

6. This problem lets you dig a bit deeper in the concept of magnetic energy (§1.9).
   (a) Starting from the induction equation, fill in the missing mathematical steps leading to eq. (1.87).
   (b) Show that in the absence of induction (meaning $\mathbf{u} = 0$), a force-free magnetic contained in a domain V will always decay.
   (c) Making use of eq. (1.91), obtain an expression involving $\mathbf{B}$ and $\mathbf{u}$ but not $\mathbf{E}$, for the Poynting flux component normal to the boundary S enclosing an electrically conducting fluid. Give a physical interpretation for each term in the resulting mathematical expression.
7. In this one you get to calculate the propagation speed of a pure Alfvén wave in an homogeneous, incompressible medium (constant density ρ). Sounds imposing, but a careful choice of geometry will make it easier than you may think. Working in cartesian geometry, consider a uniform x-directed magnetic field subjected to a velocity perturbation $u_y(x,t)$ oriented in the y direction (the problem is invariant with respect to z).
   (a) As warmup, begin by showing that eqs. (1.79) and eq. (1.83) are automatically satisfied;
   (b) then show that the ideal form of the (incompressible) MHD equations reduces to two non-trivial coupled equations, for the y-components of the flow and magnetic field;
   (c) now manipulate these two equations into a single partial second-order partial differential equation;
   (d) Show that this equation admits wave-type solutions, with phase speed given by eq. (1.86).

Bibliography:

There are a great many textbooks available on classical hydrodynamics. The following are my own top-three personal favorites:

Tritton, D.J., *Physical Fluid Dynamics*, 2nd ed., Oxford University Press (1988),
Acheson, D.J., *Elementary Fluid Dynamics*, Clarendon Press (1990)
Landau, L., et Lifschitz, E. 1959, *Fluid Mechanics*, Oxford: Pergamon Press.

If you are looking for an introduction to the topic targeted at physics-trained readers, you may want to work through the first six chapters of my class notes for PHY-3140, the 2007 version of which being still available on the Web:

`http://www.astro.umontreal.ca/∼paulchar/phy3140/notes07.pdf`

(I'm working on a revised version for the winter of 2009). If you need a refresher on undergraduate electromagnetism, you should go back to

Griffith, D.J., *Introduction to Electrodynamics*, 3rd ed., Prentice Hall (1999).

At the graduate level, the standard reference remains

Jackson, J.D., *Classical Electrodynamics*, 2nd ed., John Wiley & Sons (1975),

who does devote a chapter to magnetohydrodynamics, including a discussion of magnetic wave modes. My personal favorite on magnetohydrodynamics is:

Davidson, P.A., *An Introduction to Magnetohydrodynamics*, Cambridge University Press (2001).

Sections 1.5 and 1.10 are strongly inspired by Davidson's own presentation of the subject. He also presents an illuminating proof of the kinematic theorem embodied in eq. (1.103). The following textbook is also well worth consulting:

Goedbloed, H., & Poedts, S., *Principles of Magnetohydrodynamics*, Cambridge University Press (2004).

These authors put greater emphasis on MHD waves, shocks, and on the intersection of MHD and plasma physics. For those seeking even more focus on plasma physics aspects, I would recommend:

Kulsrud, R.M., *Plasma Physics for Astrophysics*, Princeton University Press (2005).

Also noteworthy in the general astrophysical context:

Choudhuri, A.R., *The Physics of Fluids and Plasmas*, Cambridge University Press (1998).

On Aly's theorem, see

Aly, J.-J. 1984, *Astrophys. J.*, **283**, 349,
Aly, J.-J. 1991, *Astrophys. J.*, **375**, L61,
Smith, D.F., & Low, B.C. 1993, *Astrophys. J.*, **410**, 412,

but brace yourself for some serious math.

Chapter 2

Magnetic fields in astrophysics

By now you may think you have landed in some sort of deranged combined crash course on fluid mechanics, electromagnetism... and vector algebra! To dispel this idea we now return closer to our subject matter, by briefly documenting the omnipresence of magnetic fields throughout the universe (§§2.1—2.8), pondering as to the conspicuous absence of electric fields (§2.9), and to the ultimate origin of magnetic fields (§2.10).

2.1 Earth's magnetic field

Natural magnetism (in technical parlance, ferromagnetism) is known at least since Antiquity, but it took the monumental treatise *De Magnete*, published in 1600 by William Gilbert (1544-1603), to really drive home the point that the Earth is one huge spherically-shaped bar magnet. Gilbert arrived at this conclusion from comparing the known behavior of compass needless to what he observed around a bar magnet carved into a sphere (see Figure 2.1). A medical doctor by training, in his book Gilbert also debunked many semi-occult beliefs about the behavior of magnetic objects and their influence on the human body and psyche.

To a good first approximation, the Earth's magnetic field has the form of a dipole approximately aligned with the Earth's rotation axis, with an average surface field strength of $\sim 50\mu$T. Geologic evidence has shown that the Earth's magnetic field is not steady, but flips polarities between the N and S hemisphere, these reversals being rapid (on geological timescales; they last some 10,000yr), are irregularly spaced, and punctuating much longer epochs of more or less stable field configuration, lasting a few 10^5 yr on average. At the present Earth's dipole moment is $M_\oplus = 8.1 \times 10^{22}$ A m^2. Paleomagnetic studies indicate that $M_\oplus$ has been declining rather rapidly over the past few 1000 yr, suggesting that we may be heading for a polarity reversal sometimes in the next few 1000 yr if the current trend persists.

Because the Earth's crust and troposphere are such lousy electrical conductors, the presence of the geomagnetic field is seldom felt in our daily life (and is ever more fading from popular consciousness with the replacement of magnetic compasses by GPS). In the Earth's ionosphere, however, the geomagnetic field is quite significant, and it interaction with the solar wind (to be encountered in part II of these notes) is what defines the **magnetosphere**, which happens to shield us from a lot of high energy particles often accelerated as a side effect of solar eruptive events (more on those shortly!). The impact of solar ejecta on the magnetosphere triggers geomagnetic storms. Their most spectacular manifestation being auroral emission, but the induced electric fields can pose threats to technological infrastructures such as power lines and pipelines.

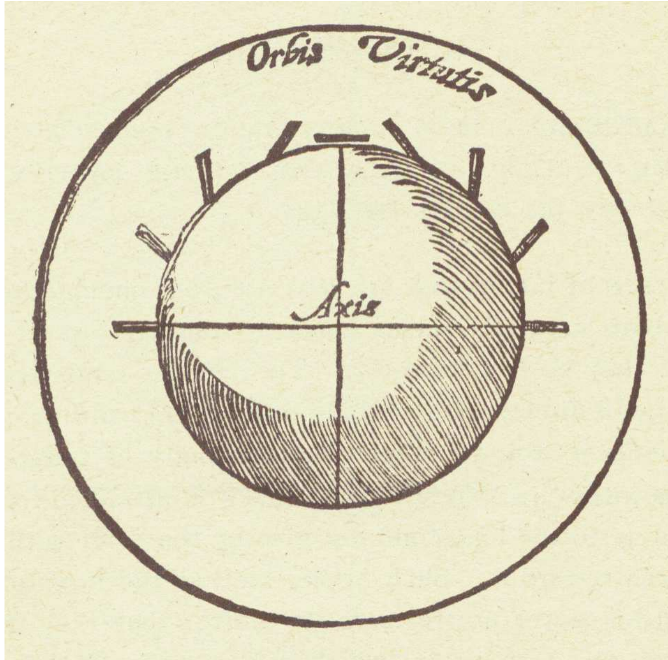

Figure 2.1:). Drawing in William Gilbert *De Magnete*, written in 1600. Gilbert polished a magnet in the form of a sphere, and could show that the pattern of inclination of the magnetic needle of as compass placed at various locations around the sphere was identical to what had already been observed by long-distance navigators and travellers of the sixteenth century.

2.2 Other solar system planets

Magnetic fields have been measured on most solar system planets (and many of the larger moons) by various space probes and landers. Table 2.1 lists some of the salient characteristics of planetary magnetic fields. Venus is the only planet in which no sign of a large-scale magnetic field has ever been detected (Pluto remains *terra incognita* as far as magnetic fields go). Given what is known of planetary internal structure, only in a few cases (Mercury, Mars) can the magnetic field be assumed to arise from ferromagnetism, in other words a "frozen-in" relic of the formation of the solar system. For all other planets, a dynamo mechanism (part III of this course) must be invoked.

Table 2.1
Planetary magnetic fields and related data

Planet	Radius [km]	Spin period [hr]	Dipole $M/M_\oplus$	Incl.[deg.]
Mercury	2400	1406	5×10^{-4}	+14.0
Venus	6100	5832	$< 10^{-5}$	N/A
Earth	6378	24.0	1	+11.3
Mars	3400	24.7	3×10^{-4}	N/A
Jupiter	71400	9.9	20000	-9.6
Saturn	60300	10.7	600	0
Uranus	25600	17.2	50	-59
Neptune	24800	16.	25	-47

The symmetry axis of the dipolar component of most planetary magnetic fields is usually inclined significantly with respect to the rotation axis, Saturn being an interesting exception to which we shall return in due time. Table 2.1 also illustrates a noteworthy trend, namely the tendency for magnetic fields to become stronger with increasing rotation rate, Mars being here the outstanding exception.

Because they have magnetic fields, solar system planets (again Venus excepted) also have magnetospheres, whose presence is beautifully confirmed by observations of auroral emission in the ultraviolet (see Figure 2.2). Jupiter's magnetosphere is particularly interesting. Besides being the biggest "object" in the solar system (Sun included), it can interact with ionized plasma ejected by volcanic eruptions on Jupiter's moon Io to drive intense electrical current systems by a dynamo process not at all unlike those we will investigate in part III of this course.

2.3 The Sun

The Sun is the first astronomical object (Earth excluded) in which a magnetic field was detected, through the epoch-making work of George Ellery Hale (1868-1938) and collaborators, in the opening decades of the twentieth century. In 1907-1908, by measuring the Zeeman splitting in magnetically sensitive lines in the spectra of **sunspots** and detecting the polarization of the split spectral components (see Fig. 2.3), Hale provided the first unambiguous and quantitative demonstration that sunspots are the seat of strong magnetic fields. Not only was this the first detection of a magnetic field outside the Earth, but the inferred magnetic field strength, 0.3T, turned out a few thousand times greater than the Earth's own magnetic field. It was subsequently realized that the Lorentz force associated with such strong magnetic fields would also impede convective energy transport from below, and therefore lead naturally to the lower temperatures observed within the sunspots, as compared to the surrounding photosphere.

The solar surface magnetic field outside of sunspots, although of much weaker strength, is accessible to direct observations, usually by measuring Zeeman broadening of spectral lines, or the degree of linear or circular polarisation of light emitted from the solar photosphere. The first magnetic maps (**magnetograms**) of the solar disk were obtained in the late 1950's by the father-and-son team of Harold D. Babcock (1882-1986) and Horace W. Babcock (1912-2003), and were little more than photographs of stacks of a few dozen horizontal scans of the solar disk displayed on an oscilloscope. Figure 2.4 (top) is a modern equivalent in pixel form, with the gray scale coding the strength of the normal component of the magnetic field (mid-level gray, $|\mathbf{B}| \lesssim 1\,\mathrm{mT}$; going to white for positive normal field, and to black for negative, peaking around 0.4T in both cases). Comparison with a continuum image (bottom) reveals that the stronger magnetic fields coincide with sunspots, but hefty fields of a few 10^{-2} tesla can be found within and around groups of sunspots, as well as in the form of small clumps anywhere else in the photosphere. Far from taking the form of a large-scale, smooth diffuse field as on the Earth, the solar photospheric magnetic field is very fragmented and topologically complex, and shows up concentrated in small magnetized regions separated by field-free plasma. This dichotomy persists down to the smallest spatial scales than can be resolved with current observational techniques. It owes much to the fact that the outer 30% in radius of the Sun is a fluid in a strongly turbulent state.

Because a fraction the solar magnetic field extends into the corona, and because it is dynamically significant there, the equilibrium structure of the corona ends up being defined by a balance between three forces: gravity, plasma pressure, and the Lorentz force. As the photospheric magnetic field inexorably evolves as a result of advection by flows and magnetic flux emergence, this equilibrium is eventually lost, leading to rapid and often spectacular disruptions of coronal structures. The associated phenomena are grouped under the general name of **solar activity**, and include phenomena as diverse as flares (Fig. 2.5) and coronal mass ejections (Fig. 2.6). The sun's magnetic field is in fact the primary energy source for the majority of such coronal transients. Saturated as we have become with spectacular images and movies from

Figure 2.2: Auroral emission observed on Jupiter and Saturn by the ultraviolet camera on the Hubble Space Telescope. Public domain images courtesy of NASA.

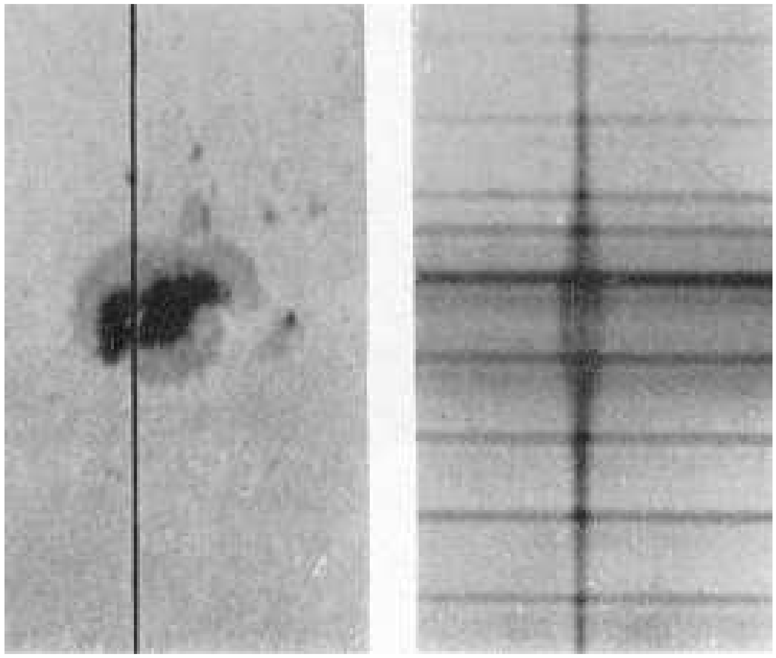

Figure 2.3: The magnetically-induced Zeeman splitting in the spectrum of a sunspot. The vertical dark line on the left image is the slit having produced the vertical stack of spectra on the right image (with wavelength running horizontally). Reproduced from the 1919 paper by G.E. Hale, F. Ellerman, S.B. Nicholson, and A.H. Joy (in *The Astrophysical Journal*, vol. **49**, pps. 153-178).

space-borne solar observing instruments, it is perhaps worth recalling that it took the best part of the twentieth century to establish the causal link between these phenomena and the solar magnetic field, and that it is really only in the mid-1970's, with the X-Ray imager and coronagraph onboard NASA's Skylab , that the coronal *terra incognita* began to be systematically explored.

There is much, much more to be said about the solar magnetic field, its spatiotemporal evolution, and its dynamical impact on the sun's photosphere and extended outer atmosphere. The most prominent temporal variations are certainly those associated with the **solar magnetic activity cycle**, which modulates, on an approximately 11-yr timescale, nearly every solar observable: coronal structures, sunspot coverage, polar field strength, radio emission, irradiance, UV and X-Ray emission, and so on, as well as the frequency of solar eruptive events (flares, coronal mass ejections, eruptive prominences, etc.). We will come back to all of this in chapter 6, but for now we leave the solar system to continue our grand tour of astrophysical magnetic fields.

2.4 Sun-like stars

The disk of solar-type stars other than the sun cannot be spatially resolved, and so direct observation of starspots is not possible, although rotational modulations of the luminosity associated with starspot darkening most certainly is. Direct measurements of magnetic polarisation of starlight is difficult as well, unless the field has a strong large-scale component, otherwise the polarisation associated with regions of opposite polarities —e.g., starspot pairs— cancel out when integrated over the solar disk. Most evidence for the presence of magnetic fields on such stars is thus indirect, yet extremely compelling, as it covers a wide range of phenomena visible

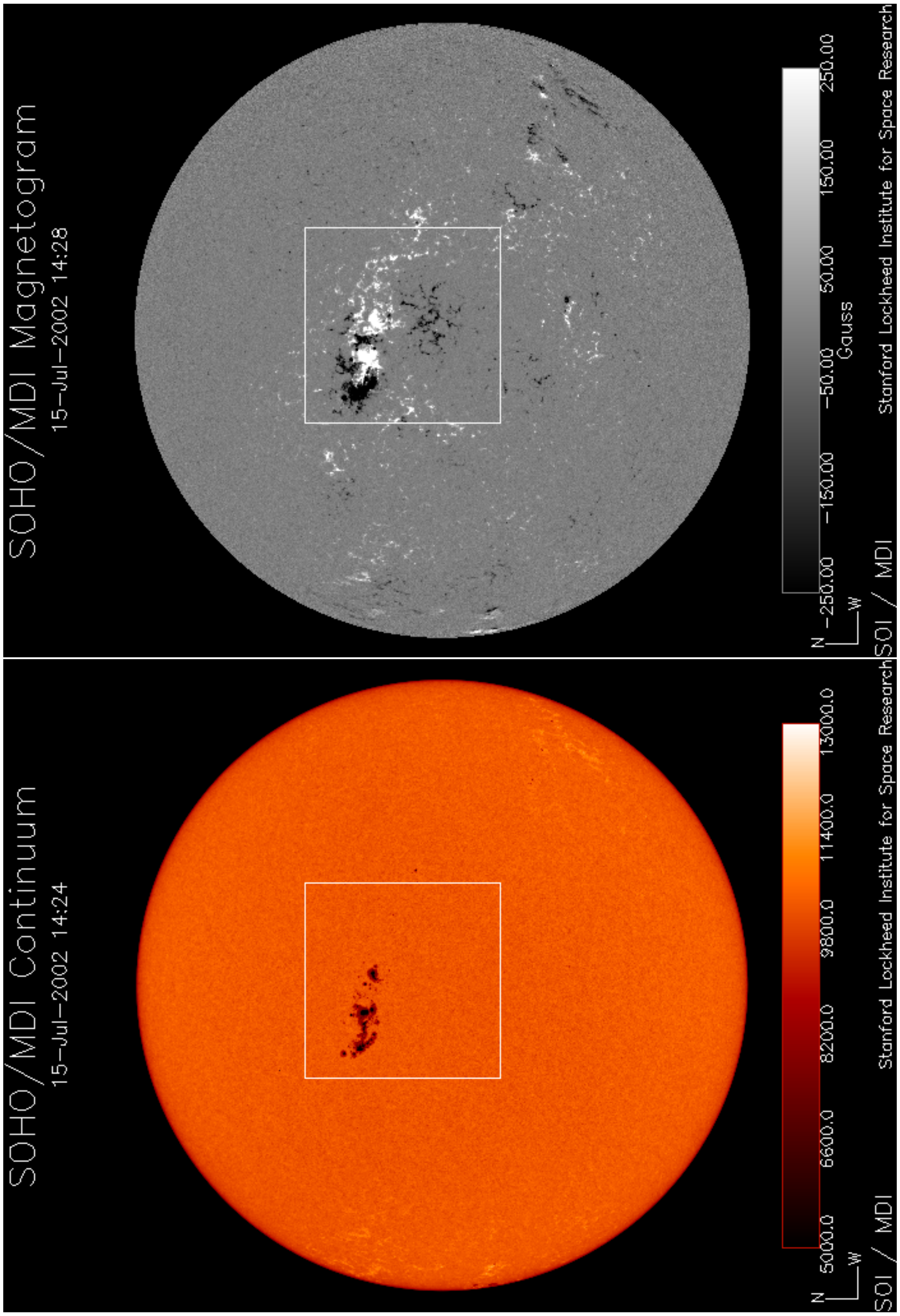

Figure 2.4: A full-disk solar magnetogram of the sun (top), with corresponding continuum image showing sunspots near disk center (bottom). Tilt the page by 90 degrees to get solar North on top, where it belongs. While the strongest magnetic fields coincide with sunspots, the magnetoram also demonstrates that the solar photosphere is filled with magnetic fields of lesser intensity, all the way up to polar latitudes. Images courtesy oif SOHO/MDI consortium. SOHO is a project of international cooperation between ESA and NASA.

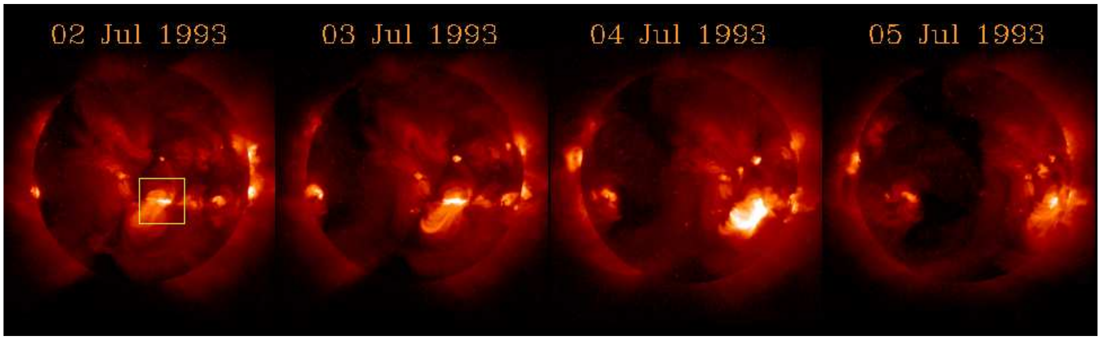

Figure 2.5: A solar flare, as seen in soft X-rays by the satellite YOHKOH. A large flare such as this one can liberate up to 10^{26} J of thermal energy in the corona over a few minutes. The bulk of that energy goes into local plasma heating and copious emission of short-wavelength radiation. Non-flaring emission of soft X-ray usually coincides with sunspots and active regions. Note also the diffuse, low level coronal X-Ray emission.

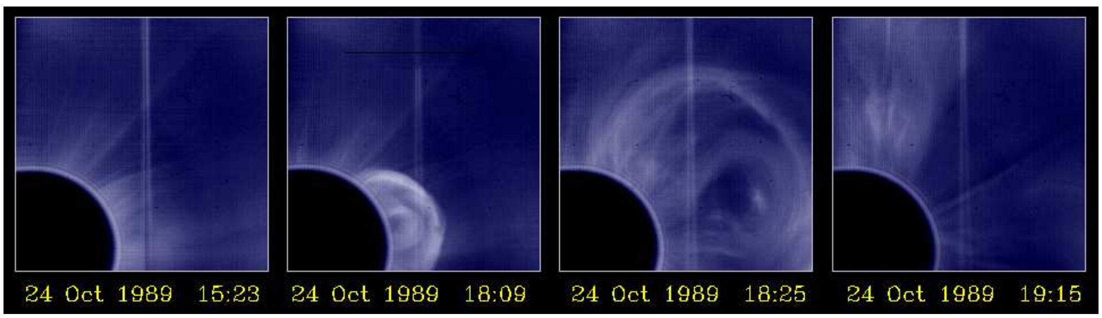

Figure 2.6: A coronal mass ejection (CME), as seen in polarized white light by the coronagraph onboard the Solar Maximum Mission satellite. Large CMEs such as this one can eject up to a few 10^9 tons of ionized plasma at speeds exceeding 10^3 km s^{-1}. The occulting disk of the coronograph, on the lower left, has a projected radius of $1.25\,R_\odot$.

on the sun, such as spectral line variability, rotational modulation of luminosity due to the passage of large starspots, flares, radio bursts, and variability in magnetically-sensitive spectral lines on a wide range of timescales.

Figure 2.7 shows a time series of X-Ray emission obtained by the ROSAT satellite, that looks very much like time series of disk-integrated flux observed by the the Earth-orbiting GOES satellites when a solar flare is taking place. The most likely interpretation of Fig. 2.7 is that ROSAT had the good fortune to catch a solar-type star just as it was producing a large flare.

Another magnetic field-related stellar observable that is particularly noteworthy is the emission in the cores of the H and K lines of CaII (396.8nm and 393.4nm, respectively). On the Sun, this emission is known to be associated with non-radiative heating of the upper atmosphere, and is known to scale well with the local photospheric magnetic flux. Starting back in 1968 at Mt Wilson Observatory, Olin C. Wilson (1909-94) began measuring the CaII H+K flux in a sample of solar-type stars, a laborious task that was later picked up by a brave group of undeterrable associates and followers, whose collective labor has produced a 40 year long archive of CaII emission time series for no less than 111 stars in the spectral type range F2-M2, on or near the main-sequence.

Figure 2.8 shows a few sample time series of the so-called Calcium index S, mesuring the ratio of core emission intensity in the H and K lines to that of the neighbouring continuum. Some stars show solar-like cycles, others have irregular CaII emission, some show long term trends and others can only be dubbed "flatliners". Note that the mere presence of detectable

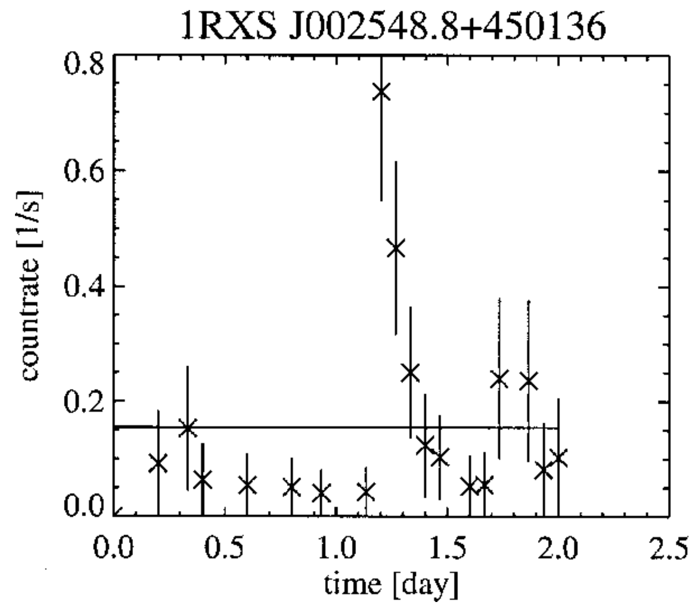

Figure 2.7: X-Ray emission from a stellar source, as observed from the ROSAT satellite. The rapid rise (minutes) and slower decay (many hours) is similar to what is observed in disk-integrated X-Ray detections of solar flares. Figure reproduced from Fuhrmeister & Schmitt 2003, A&A **403**, 247-260 [Figure 4].

CaH+K emission indicates magnetic activity; the absence of detectable temporal variations in flatliner stars simply means that they stars lack a solar-like well-organized magnetic cycle. Among cyclic stars, it was shown that a relatively tight parametric relationship exists between the cycle period ($P_{\rm cyc}$) and rotation period ($P_{\rm rot}$):

$$P_{\rm cyc} \propto \left(\frac{P_{\rm rot}}{\tau_c}\right)^{1.25} , \qquad (2.1)$$

with τ_c being the convective turnover time estimated from mixing length theory of convection. The quantity within parenthesis is related to the so-called Rossby Number, measuring the influence of the Coriolis force on a flow. As we shall see in due time, such a link between rotation, convection and cycle period is indeed expected from dynamo theory. Later studies have shown that eq. (2.1) is probably an oversimplification, and will return to these remarkable data in part III of the course, when we construct dynamo models for the sun and stars.

The important conclusion here is that the Sun is not some weird oddball: indirect observational evidence for magnetic fields has been found on *every* late-type main-sequence star observed with sufficient sensitivity. Moreover, evidence for solar-like magnetic activity in late-type stars stops rather abruptly around spectral type F0-F2 on the main-sequence, which, according to current stellar structural models, coincides with the disappearance of significant surface convection zones.

2.5 Early-type stars

Although most main-sequence stars seem to have gone "magnetically quiet" on the hot side of the dividing line at F0-F2, extant observations suggest a true dichotomy with regards to stellar magnetism in intermediate-mass stars: most A and B stars (around 95%) on or near the main-sequence have no measurable magnetic field, but nearly all those who do combine strong, large-scale magnetic fields, steady on decadal timescales at least, with slow rotation

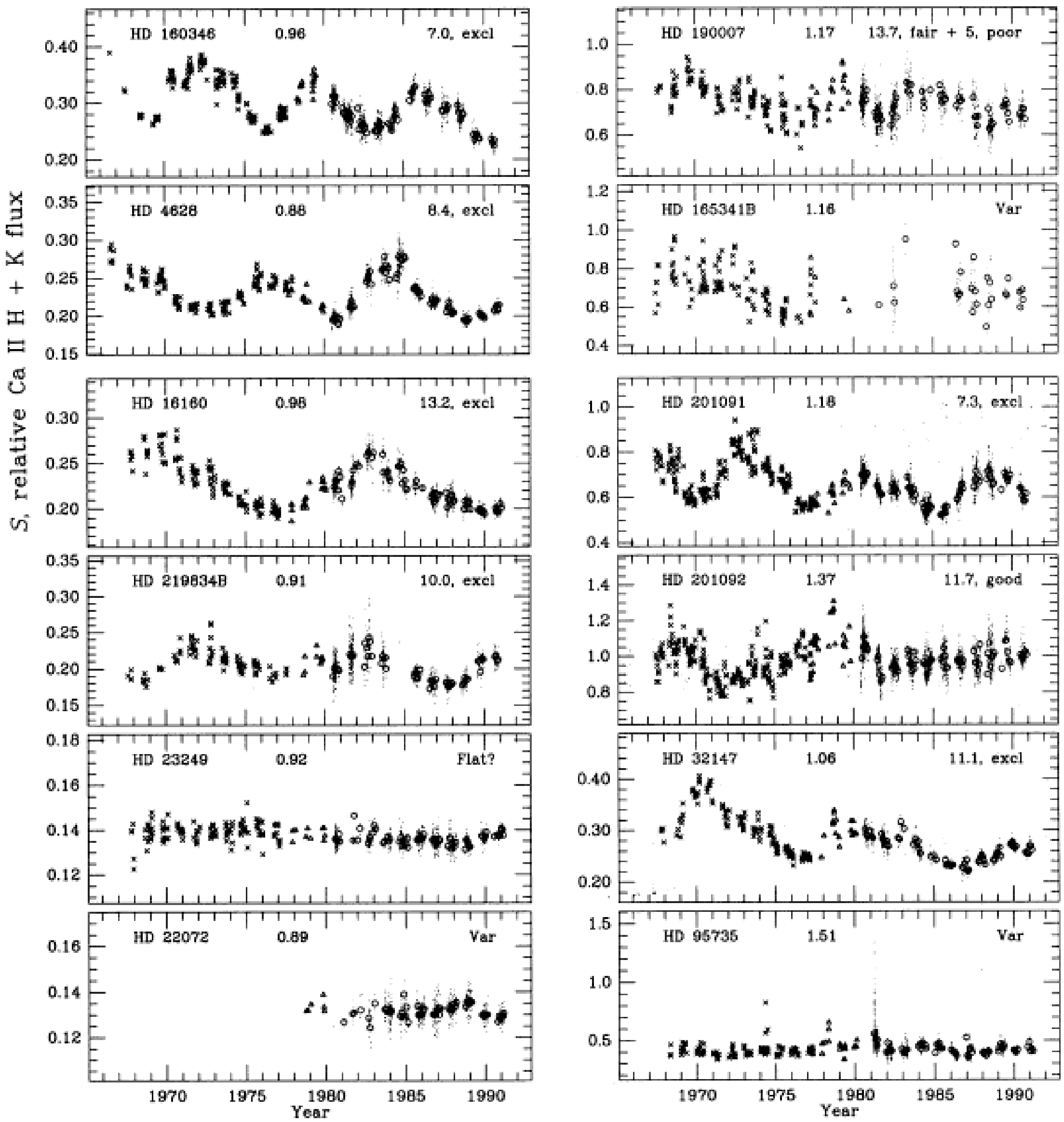

Figure 2.8: Calcium emission index in a small subsample of the Mt Wilson dataset, showing the variety of CaII emission patterns: cycles, non-cyclic irregular emission, long term trend, and constant emission. On such plots, the sun would have a mean emission level $\langle S_\odot \rangle = 0.179$, with a min/max range of about 0.04. Figure cropped from a much larger Figure in Baliunas *et al.* 1995, ApJ,438, 269 [Figure 1g].

and pronounced photospheric abundance anomalies. As we will see later in this course, the presence of a strong, large-scale photospheric magnetic field (ot whatever origin) favors angular momentum loss, and therefore slow rotation; and a strong magnetic field and low rotation favor atmospheric stability, giving full leeway for chemical separation to operate and alter photospheric abundances.

In the most slowly rotating, strongly magnetized Ap stars, the mean surface magnetic field stength ("mean" in the sense of being averaged over the stellar surface) can be detected by Zeeman splitting, as in sunspot. Figure 2.9 below shows a striking example of such splitting. In more rapidly rotating stars magnetic Doppler imaging become a possibility; this relies on the varying shapes of spectral lines formed as magnetic structures cross the visible part of the stellar disk. "Imaging" remains indirect, in the sense that the stellar surface is of course not resolved spatially, but the availbility of many spectral lines, with some appropriate regularization scheme, allows this inverse problem to be solved. Figure 2.10 shows a particularly well-studied exemplar, namely the chemically peculiar star 49Cam. The field strength is high, the magnetic topology quite complex, with the idea of a strongly inclined dipole, historically the common interpretation for Ap stars magnetic fields, being a rather rough approximation here.

It is an intriguing fact that the few chemically-normal, (relatively) rapidly rotating early-type stars on which magnetic fields have been detected all sit in the early-B range of spectral types and belong to the βCep sub-class (and include the prototype star βCep itself). However, indirect evidence for photospheric magnetism in O and B star has been accumulating steadily, be it as emission of hard radiation above and beyond what shock dissipation can provide, channelling of stellar winds, and spectral variability. Ongoing spectropolarimetric campains targeting massive stars will hopefully provide more data for theoreticians/modellers to chew on in upcoming years.

2.6 Pre- and post-main-sequence stars

As with main-squence late-type stars, abundant evidence for magnetic fields in pre- and post-main sequence stars of spectral types later than F has now been accumulating, mostly again in the form of stellar analogs to well-observed solar phenomena: X-Ray and EUV emission, flaring, spectral variability, rotational modulation by starspots, and so on. More recently magnetic Doppler imaging has been used to reconstruct the surface magnetic field of some pre-main-sequence stars in the TTauri evolutionary phase. Whether TTauri or giants, all these stars have low surface temperature and thick convection zones, so observations of magnetic activity indicators similar to what is observed in late-type main-sequence stars points once again to the importance of convection zones of significant radial extent below the photosphere. Indeed, there seems to exist a rather clear-cut, slightly inclined dividing line bisecting the upper part of the HR diagram (main-sequence and up in luminosity), on the right side (low $T_{\rm eff}$) of which evidence of magnetic activity is ubiquitous. Things get messy again with very cool supergiants, with signs of magnetic activity disppearing across various not quite coincident dividing lines, depending on the indicator chosen (X-Ray emission, non-thermal emission lines, etc).

With classical TTauri stars, additional complications also come from the presence of an accretion disk, itself most likely magnetized, perhaps the site of magnetic field generation by dynamo action, and perhaps even magnetically coupled to its central star. Such a coupling has been invoked to explain the (relatively) low rotation rates of TTauri stars, which after all are contracting and accreting large amounts of mass —and angular momentum— from their disk, and should therefore spin up far more than is observed. Indeed, without angular momentum loss mediated by magnetic fields in the early stages of star formation, it is quite likely that stars could simply not eliminate enough angular momentum to form at all!

In hot post-main sequence stars, the observational situation is not well documented or understood. It is a remarkable fact that magnetic fields have been detected in *all* sdO and sdB hot subdwarfs for which a serious attempt has been made. The evolutionary status of these objects is not well-understood, but they most likely represent what used to be the inner

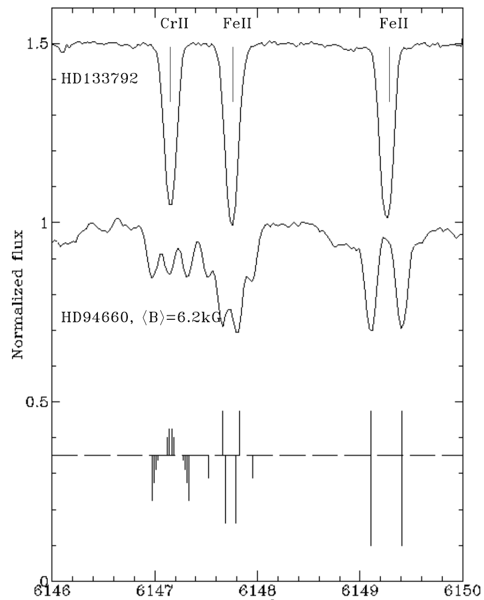

Figure 2.9: Zeeman splitting of magnetically-sensitive absorption line in the spectrum of the Ap star HD94660. The inferred mean field strength for this star is 0.62T. The top trace is that of a typical unmagnetized star of similar spectral type. The horizontal axis is the wavelength, measured in Å. Figure reproduced from the Mathys *et al.* paper cited in the bibliography, with a few labels added.

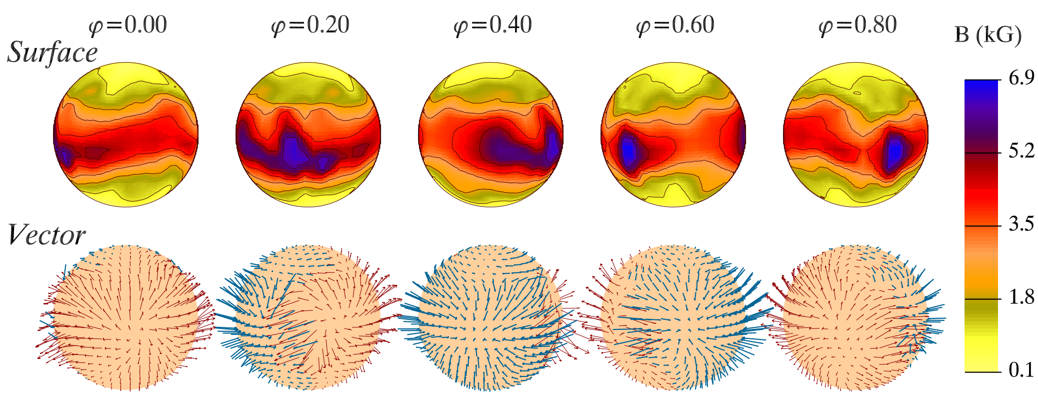

Figure 2.10: The surface magnetic field on the Ap star 49Cam, as reconstructed for various rotational phases (φ) by magnetic Doppler imaging. The top row shows the net field strength, and the bottom row the orientation of the surface magnetic field vector. Plot courtesy of J. Silvester and G. Wade, RMC/Kingston.

core of giants prior to the episode of strong mass loss that accompanies the transition to the horizontal branch. Detection of kG-strength magnetic fields in such stars is strong evidence for the existence of magnetic fields in the deep interior of their main-sequence progenitors.

2.7 Compact objects

Magnetic fields in isolated white dwarfs have been detected through circular polarisation measurements in the wings of strong spectral lines, usually Balmer lines in the so-called "DA" white dwarfs showing Hydrogen lines in their photospheres. Actual Zeeman splitting is only detected in the most strongly magnetized objects ($\gtrsim$ a few 10^2 T). Inferred field strengths range from a few T up to a whopping 10^5 T, with the overall incidence of magnetism standing at a few percent. However, these techniques are only sensitive to large-scale magnetic fields, still producing a net polarisation signal when integrated over the stellar disk, and so the true incidence of magnetism in white dwarfs may actually be significantly higher.

Inferred magnetic field strengths in neutron stars range from 10^4 to 10^{11} T. Neutron stars magnetic fields are of course most readily detected via the pulsar phenomenon, most likely arising from slight misalignement of the magnetic axis with respect to the rotation axis of the (very rapidly rotating) neutron star. It is quite striking that the highest strengths of large-scale magnetic fields in main-sequence stars (a few T in Ap stars), in white dwarfs ($\sim 10^5$ T) and in the most strongly magnetized neutron stars ($\sim 10^{11}$ T) all amount to similar surface magnetic fluxes, lending support to the idea that these high field strengths can be understood from simple flux-freezing arguments (§1.10... and Problem 2.1!). There is also observational evidence that actual magnetic field evolution is taking place as pulsars age, but this remains very slippery territory, both from the modelling and observational points of view.

Observationally, very little is known about black holes except that there is quite possibly one at the center of our galaxy, so you won't be surprised to hear that even less is known about black hole magnetic fields. One should perhaps just point out that solutions to the field equations of general relativity for rotating, electrically charged black holes do exist, which is a good start towards magnetic fields production. Evidence to date is limited to energetic phenomena interpreted in terms of magnetic channelling of material onto the black hole. But beyond that, at the present time there is only religious fervor.

2.8 Galaxies and beyond

Magnetic fields in the diffuse, low-density interstellar gas is most readily detected through synchrotron radiation emitted by relativistic charged particles spiralling along magnetic fieldlines. This technique is succesfull not only within the Milky Way, but also for other galaxies. Other means of detection, for the time being limited to the Milky Way, include the polarisation of optical starlight by elongated (i.e., non-spherical) dust grains aligning themselves perpendicularly to magnetic fieldlines; these aligned dust grains also sometimes emit detectable polarized infrared radiation. Finally, for relatively strong fields Zeeman splitting of spectral lines in the radio domain has also been measured. As with stars, magnetic fields seem to be ubiquitous features in pretty much all galaxies.

The galactic magnetic field in the solar neighbourhood has a strength of about 0.6nT, up to a few nT near galactic center. This is indeed typical of spiral galaxies, which show field strengths in the range 0.5–1.5nT, up to some 3nT in high density regions of spiral arms. The strongest large-scale galactic magnetic fields so far measured have strength reaching 1nT, and have been found in starburst galaxies. While this may seem quite low values, such field strengths have important consequences for star formation, the distribution of cosmic rays, and equilibrating the interstellar medium against gravity.

Given that most stars appear to be magnetized to some degrees, and that many stars tend lose mass (some by blowing up!), it is perhaps not surprising to detect magnetic field in the galactic interstellar medium. What is surprising is that this magnetic field tends to be organized on large spatial scales, commensurate in fact with galactic dimensions. An example is shown on Figure 2.11, showing radio intensity isocontours and polarisation vectors superimposed on an optical image of the spiral galaxy M51. Such large-scale, spatially well-organized magnetic fields are most likely produced by a dynamo mechanism, not at all dissimilar to that responsible for the presence of magnetic fields in many stars, including the Sun. Additional, indirect evidence for well-organized large-scale magnetic fields in galaxies include the collimation of jets, and energetic phenomena often encountered in quasars and AGN; at the present time, the most convincing physical models for such phenomena all involve magnetic fields at some level.

Indirect evidence for the existence of extragalactic magnetic fields exists, with an upper limit of $\sim 10^{-3}\,$nT on the mean field strength over length scales of order 100MPc and larger. These fields could be primordial in origin, or could have been ejected in intergalactic space by galactic winds.

2.9 Why B and not E?

Even the very brief survey of astrophysical magnetic fields of the preceding section should have made it clear that there are magnetic fields of all strengths and shapes pretty much everywhere we look in the known universe. Yet electric fields are conspicuously absents. Why is that? You might think, looking at Maxwell's equations (1.52)–(1.55) that $\mathbf{E}$ and $\mathbf{B}$ appear therein on apparently equal footing, leaving nothing to allow us to anticipate the observed astrophysical preponderance of magnetic fields over electrical fields. Moreover, one observer's magnetic field can be turned into another's electric field by judicious change of reference frame. So what's the deal here?

Well, for one thing if you use any sort of sensible "rest frame" for astronomical observation (Earth at rest; solar system at rest; Milky Way at rest; local group at rest; etc *ad infinitum*) there is a lot of $\mathbf{B}$ around and precious little $\mathbf{E}$. The crucial difference between $\mathbf{E}$ and $\mathbf{B}$ in Maxwell's equations is not the fields themselves, or the reference frame in which they are measured, but in their *sources*. The Universe may be largely empty, but the fact is that is contains a whopping number of electrically charged particles of various sorts (free electrons, ionized atoms or molecules, photoelectrically charged dust grains, etc). If a large-scale electric field were suddenly to be turned on, all these charges will do the honorable thing, which is to separate along the electric field direction until the secondary electric field so produced cancels the externally applied electric field, at which point charge separation ceases. Moreover, the low densities of most astrophysical plasmas lead to very large mean-free paths for microscopic constituents, leading in turn to fairly good electrical conductivities and very short electrostatic relaxation times τ_e (see eq. (1.71)), even when the ionisation fraction is quite low (such as in molecular clouds). In other words, astrophysical electric fields, if and whenever they appear, get shortcircuited mighty fast.

Not so with magnetic fields. For starters, as far as anyone can tell there are no magnetic monopoles out there (well, maybe just one, of primordial origin... more on this shortly), so shortcircuiting the magnetic field by monopole separation is out of the question. Magnetic fields, left to themselves, will simply decay as the electrical currents that support them (remember Ampère's Law) suffer good ol'Ohmic dissipation. We already obtained a timescale for this process given by eq. (1.64), and we already noted, on the basis of the compilation presented in Table 1.2, that this timescale is extremely large, often exceeding the age of the universe. Once magnetic fields are produced, by whatever means, they stick around for a long, long time.

2.10 The ultimate origin of astrophysical magnetic fields

So, there are magnetic fields all over the place in the Universe. How did they originate? If we stick to MHD, then we immediately hit a Big Problem, arising from the linearity of the MHD induction equation (1.60): if $\mathbf{B} = 0$ at some time t_0 then $\mathbf{B} = 0$ at all subsequent times $t > t_0$, a problem that persists unabated as t_0 is pushed all the way back to the Big Bang.

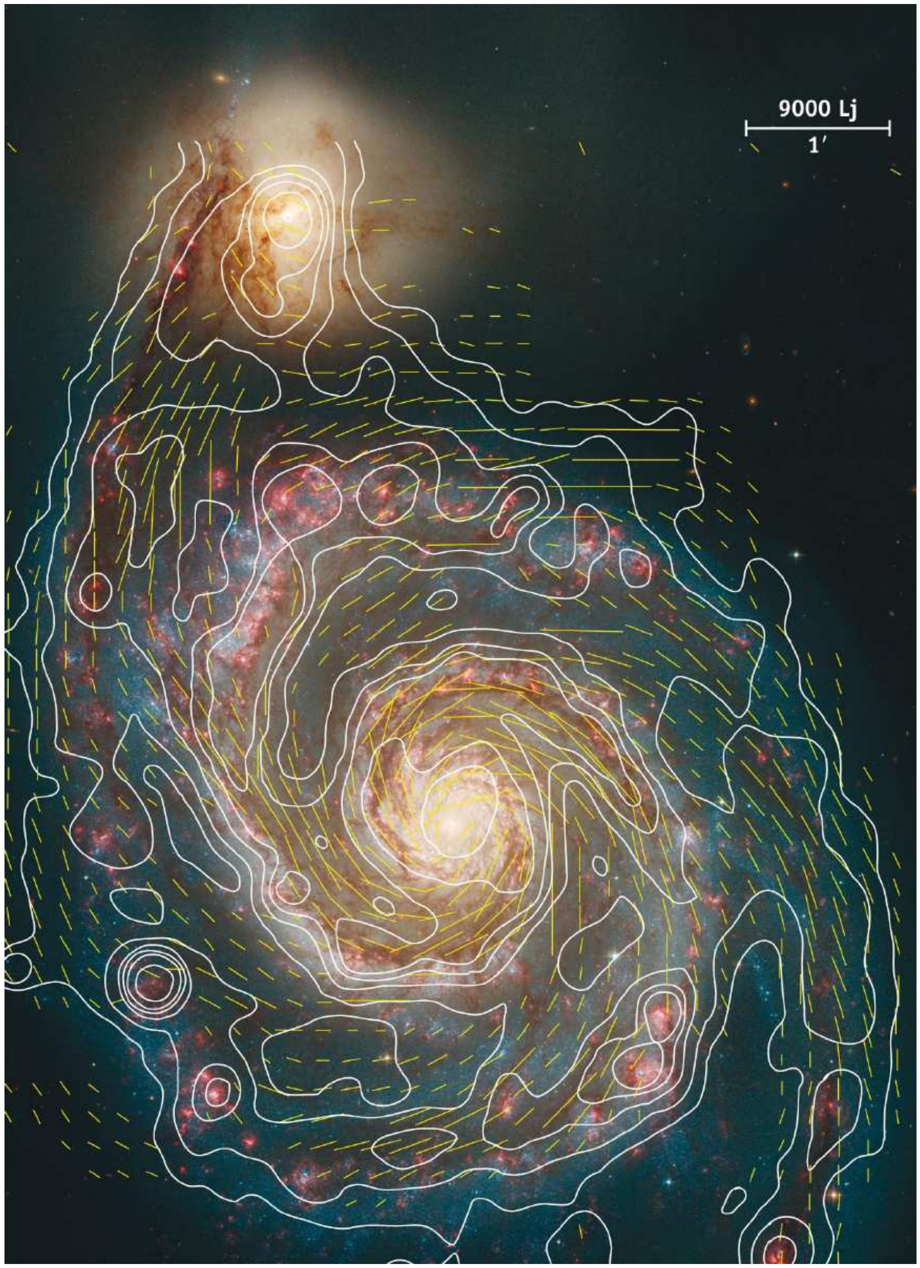

Figure 2.11: Optical image (Hubble) with overlaid isocontours of radio emission intensity at $\lambda = 6\,\mathrm{cm}$ (in white) and polarisation orientation (orange line segments, both from VLA observations). Note the large-scale organization of the magnetic field, following the optical spiral structure. Image downloaded from the *Scholarpedia* article by Rainer Beck cited in the bibliography at the end if this chapter.

In part III of this course we will see that astrophysical flows are actually quite apt at amplifying magnetic fields, so what we are after here is a very small "seed field" to start up the process. Cheap and easy explanations along the line of an original seed magnetic field being a primordial relic of the Big Bang need not concern us here. Nor is early-universe ferromagnetism a viable option, since permanent magnets require an externally-applied magnetic field to become magnetized in the first place. Interestingly, the two options that are currently deemed viable stand at the opposite ends of the physical exotism scale: magnetic monopoles... and batteries. Let's briefly discuss these in turn[1].

2.10.1 Magnetic monopoles

On the theoretical front, already back in 1931 Paul Dirac (1902-1984) pointed out that there is nothing to prevent there being magnetic monopoles so long as the magnetic charge on a particle is some integer multiple of $g \equiv hc/(4\pi e) \approx 69e$, where h is Planck's constant, and e is the fundamental electric charge. With just *one* magnetic monopole in the universe we have our basic seed field. In the early 1970's, G. t'Hooft and A.M. Polyakov argued that the spontaneous symmetry-breaking of the Grand Unified (field) Theory Lagrangian, which occurs very early in the formation of the universe at $k_BT \approx 10^{15}$ GeV, would produce a *lot* of $m_g \approx 10^{16}$ GeV/c^2 magnetic monopoles.[2] So many in fact that inflationary cosmology was invented in part to deal with this embarrassment of riches and to leave about one monopole within each subdomain of the inflated universe(s). But again, we only need one monopole to produce our seed field, so the realist stops there.

2.10.2 Batteries

Leaving magnetic monopoles aside, we should inquire about more pedestrian means to create seed magnetic fields. Since it could be that t'Hooft and Polyakov got the wrong Lagrangian, GUT's will be superseded by something else, etc. So it would be nice to have a fall back mechanism to generate a seed magnetic field that relies on basic physics that we know functions sensibly at least in our part of the universe. To this end, we return to our derivation of the induction equation (§1.3). Recall that one essential next step toward MHD from Maxwell required stipulating Ohm's law, in the form of eq. (1.58) for the laboratory frame of reference. Consider now the possibility of a "mechanically-driven" process of charge separation (i.e., not related to the presence of an electric field in any reference frame); Ohm's law then picks up an extra term:

$$\mathbf{J} = \sigma\Big[\mathbf{E} + \mathbf{U}\times\mathbf{B}\Big] + \mathbf{J}_{\rm mech} \ . \tag{2.2}$$

If we keep only the very first term on the RHS of equation (2.2), *and* drop the displacement current in equation (1.55), then we get back to the induction equation (1.60). If we avail ourselves of neither of these opportunities then we obtain instead:

$$\left\{1 + \frac{\eta}{c^2}\frac{\partial}{\partial t}\right\}\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\Big(\mathbf{U}\times\mathbf{B} - \eta\nabla\times\mathbf{B} + \mu_0\mathbf{J}_{\rm mech}\Big) \ . \tag{2.3}$$

Notice that our only hope for creating $\mathbf{B}$ out of nothing (so to speak) is the $\mathbf{J}_{\rm mech}$ term; retaining the displacement current gives us no advantage.

The $\mathbf{J}_{\rm mech}$ term represents our ability to mechanically grab a hold of electric charges and force currents to flow; in other words, an electromotive force. In the dense interior of a conducting star, plasma kinetic theory permits one to write down a prescription for this "battery" contribution to the total electric current density as:

$$\mathbf{J}_{\rm mech} = \frac{\sigma}{e n_e}\Big[\nabla p_e - \frac{1}{c}\mathbf{J}\times\mathbf{B}\Big] \ , \tag{2.4}$$

[1]Most of the remainder of this section was written by T.J. Bogdan as part of an earlier version of these class notes, with only some slight rewording, notational homogeneization, and minor additions on my part.

[2]You might find it amusing to figure out how many kilograms that works out to be!

where p_e is the contribution of the electrons alone to the thermal pressure (see references in bibliography). For a completely ionized pure hydrogen plasma, p_e is just half of the total gas pressure, and $n_e = \rho/m_p$, and so,

$$\mathbf{J}_{\rm mech} = \frac{\sigma m_p}{2e\rho}\left[\nabla p - \frac{2}{c}\mathbf{J}\times\mathbf{B}\right] . \tag{2.5}$$

Now, the second term on the RHS of equation (2.5) does not do us any good since it carries a factor of **B**, so the whole plan rests upon the first term generating a seed magnetic field. For a spherically symmetric star, we know from hydrostatic equilibrium that $\nabla\Phi = (\nabla p)/\rho$, and so the product $\eta\mathbf{J}_{\rm mech} \propto \nabla\Phi$. Which does not do us any good because of the presence of the curl operator on the RHS of equation (2.3), which yields zero upon acting on $\mathbf{J}_{\rm mech}$ since (∇p) is a gradient of a scalar function. How can we get around this constraint? A viable possibility is rotation. If a star is rotating, then there is a centrifugal force per unit density of $\varpi\Omega^2\hat{\mathbf{e}}_\varpi$ which adds to $\nabla\Phi$ and which leads to the generation of a seed magnetic field. This process of the centrifugal force driving a flow of electrons relative to the ions was first pointed out by Ludwig Biermann (1950) and is now called the *Biermann battery*.

In fact *any* process that can produce a relative motion between the ions and electrons is a potential battery mechanism, and a possible candidate for creating seed magnetic fields. For example, consider a rotating proto-galaxy, where the outer portions of the proto-galaxy move at a speed $U = R\Omega$ relative to the frame in which the microwave background is isotropic. The Thomson scattering of the microwave photons by the electrons results in the so-called *Compton drag effect*, which causes the electrons to counter-rotate with respect to the ions The net result is an azimuthal current which generates a poloidal magnetic field.

Of course, if you bother to put typical numbers in these various examples you will find that you don't really generate very *much* magnetic field. But generating a lot of field is not the point, that can be one via magnetic flux conservation in a collapsing protostellar cloud, or, as we shall see in part III of this course, via the $\mathbf{u}\times\mathbf{B}$ term in our MHD induction equation. The basic idea to take away from this section is that invoking weird, unproven physics to get away from $\mathbf{B} = 0$ is not necessary.

Time to move on to part II of this course; for the time being, we just consider that most stars are magnetized to some degree, and investigate in the following 3 chapters the impact such magnetic fields may have on mass loss via wind-like outflows.

Problems:

1. This problem lets you explore some astrophysical implications of the flux freezing constraints encountered in the previous chapter.

   (a) Assume that the Sun has formed from the spherically symmetric collapse of an initially spherical gas cloud of radius 10 light-years and threaded by a large-scale galactic magnetic field of strength 10^{-10} T. Under the assumption of flux freezing, what should then be the strength of the internal solar magnetic field? Is this a reasonable number?

   (b) The large-scale surface poloidal field of the Sun is actually of order 10^{-3} T. Under the same assumptions as in (a), compute the strength of the magnetic field expected in a White Dwarf ($R_{\rm WD}/R_\odot = 0.01$). Is this a reasonable number?

   (c) What would you think are the primary problem with these two estimates?

2. Let's consider a constant-density "sun" made of purely ionized Hydrogen. Suppose now that its *exterior magnetic* field can be approximated by a dipole, with a surface field strength of 10^{-3} T. Assume now that this magnetic field is produced by an azimuthal (i.e., ϕ-directed) current density within the interior ($r/R_\odot < 1$); then,

(a) Estimate the magnitude of the current density required to produce such a dipolar field;

(b) Estimate the drift velocity between protons and electrons required to produce such a current density. How does it compare to the average thermal velocity?

(c) How can such a small velocity difference not be erased by collisions between microscopic constituents? To answer this one will have to think back to some fundamental aspects of the induction process, as covered in your first course on electromagnetism.

3. Suppose that a large flare, liberating some $10^{26}\,J$, is power by the complete dissipation of a uniform magnetic field of strength 0.1 tesla contained within a volume V. How large does V has to be to add up to 10^{26} J? Does this make sense, given typical dimensions for a large active region (which you can eyeball, e.g., from Fig. 2.4).

4. With this one you get to play with some of the timescales introduced in the preceding pages. We'll consider a solar-like star, with uniform electrical conductivity throughout (feel free to use the value in Table 1.2.

   (a) Estimate the diffusion time for a magnetic field contained in the solar interior. How large would the magnetic diffusivity have to be to produce a typical diffusive timescale of the order of the observed solar cycle period of $\sim 10\,$yr? Do you think this is a reasonable number?

   (b) Estimate now a flow speed that would lead to a turnover time of order 10 yr. Is this a reasonable number in the solar/stellar context?

5. At long last, a problem that asks you to **surf the Web!!** Use your surfing skills to locate what is (in your opinion) a **very cool** image of some astrophysical magnetic field(s). But careful now, you must be able to explain what is actually on the image! And please provide a full working URL.

Bibliography:

Many ambitious monographs have been written on the general topic of astrophysical magnetic fields. My personal "top-three" selection is:

Parker, E.N. 1979, *Cosmic magnetic fields*, Oxford: Clarendon,
Mestel, L. 1999, *Stellar magnetism*, Oxford: Clarendon,
Rüdiger, G., & Hollerbach, R. 2004, *The magnetic universe*, New York: John Wiley.

Parker's book is unfortunately out of print, and Rüdiger & Hollerbach's outrageously priced ($230!). The Mestel book was issued in paperback in 2003, but still at a whopping $125! For good introduction on planetary magnetic fields, see

Bagenal, F. 1992, *Ann. Rev. Earth & Planet. Sci.*, **20**, 289-328

Table 2.1 was in fact adapted from this paper, augmented by data extracted from the Goedbloed & Poedts textbook cited in the preceding chapter. The following (relatively) recent conference proceedings is devoted to documenting the ubiquitous presence of magnetic fields throughout the Hertzsprung-Russell diagram:

Mathys, G., Solanki, S.K., & Wickramasinghe, D.T. (eds.), *Magnetic fields in the Hertzsprunbg-Russell diagram*, ASP Conf. Series **248** (2001).

See in particular the review papers by Mathys, Bagnulo (Ap stars), Campbell (binary stars), Valenti & Johns-Krull (cool stars), Schmidt (white dwarfs) Reisenegger (neutron stars), as well as the introductory overview paper by Mestel.

The following are two good, recent textbooks on Solar Physics at the higher undergraduate level, including extensive discussions of solar magnetism:

Stix, M., 2001, *The Sun: an introduction*, 2nd ed., Springer,
Foukal, P., 2006, *Solar Astrophysics*, 2nd ed., John Wiley & Sons.

If the historical aspects of sunspots and solar magnetic studies are of interest to you, you may also want to consult the ever-being-enlarged Web site "Great Moments in the History of Solar Physics":

http://www.astro.umontreal.ca/~paulchar/grps/histoire_physique.html.

Hale's original papers on sunspots are still well worth reading. The two key papers are:

Hale, G.E. 1908, "On the probable existence of a magnetic field in sunspots", *The Astrophysical Journal*, **28**, 315-343,
Hale, G.E., Ellerman, F., Nicholson, S.B., and Joy, A.H. 1919, *The Astrophysical Journal*, **49**, 153-178.

Figure 2.8 was taken from the following paper, still today one of the more cogent exposition of the Mt Wilson CaII project and data:

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and more specifically on Ap stars:

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On magnetic field detection in hot subdwarfs, see

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I found the following online article a good starting point to learn about galactic magnetic fields:

Beck, R. 2007, *Scholarpedia*, **2**(8), 2411
http://www.scholarpedia.org/articles/Galactic_magnetic_fields.

For some "light" reading on magnetic monopoles in field theory and astrophysics, try,

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and references therein. For more on Biermann's battery, see

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as well as chap. 13 in the Kulsrud book cited in the bibliography of the preceding chapter.