Astrophysical Radiation Magnetohydrodynamics

Being an Opera in Four Acts

ACTS I & II

Lecture 1: Basic Theory

With Words & Music By

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(aka Captain Louis Renault)
25 September 2022

<u>Renault</u>: And what in Heaven's name brought you to Montréal?

<u>Rick</u>: My health. I came to Montréal for the **Astrophysical Fluids**.

<u>Renault</u>: Fluids? What Fluids? We are frozen in winter!

Astrophysical Radiation Magnetohydrodynamics

Being an Opera in Four Acts

ACT I

All the universe is a stage,
And all matter and fields are merely players;
They have their exits and entrances,
And each, in their time, plays several parts.

Summary

Transport Coefficients

Radiation as a Relativistic Fluid

***Nano**scopic*	$i\hbar\frac{\partial}{\partial t}\vert\psi\rangle = \boldsymbol{\mathcal{H}}\vert\psi\rangle$	quantum mechanics
***Micro**scopic*	$\dot{q}_i = \frac{\partial \boldsymbol{\mathcal{H}}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \boldsymbol{\mathcal{H}}}{\partial q_i}$	dynamical systems/ classical mechanics
***Meso**scopic*	$\frac{\partial \psi}{\partial t} + \frac{1}{m}\boldsymbol{p}\cdot\frac{\partial \psi}{\partial \boldsymbol{x}} + \boldsymbol{f}\cdot\frac{\partial \psi}{\partial \boldsymbol{p}} = \frac{\delta \psi}{\delta t}$	plasma/kinetic theory radiative transfer
***Macro**scopic*	$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\rho\boldsymbol{u} = 0$	fluid/continuum mechanics
***Mondo**scopic*	$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2KE + 3P + EM + R + W$	thermodynamics/ virial theory

Summary

Energy Conservation

$$t \to t' \pm t$$

Entropy Conservation

Momentum Conservation

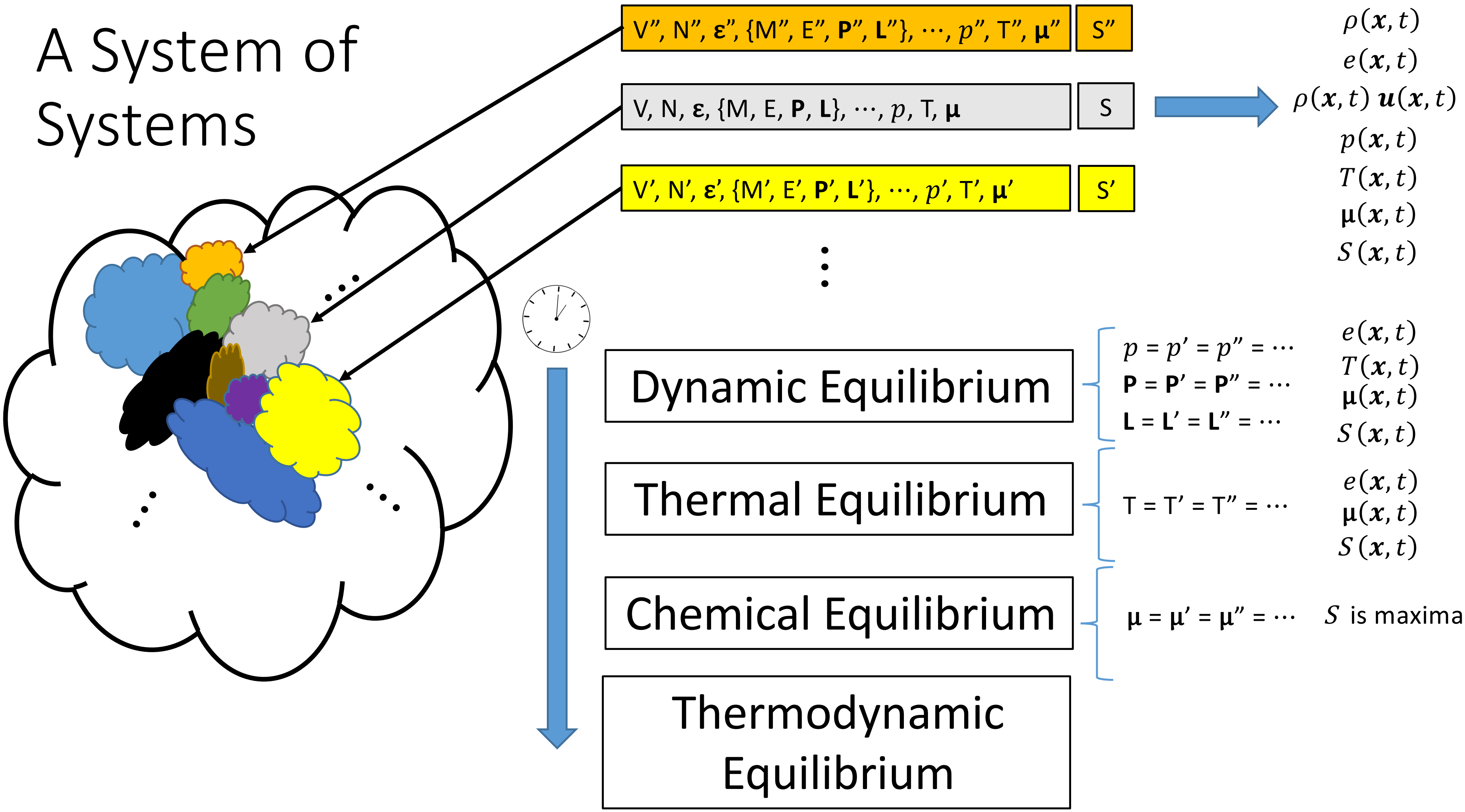
Parity Conservation

$$\boldsymbol{x} \to \boldsymbol{x}' + \mathbb{R}\boldsymbol{x}$$

Angular Momentum Conservation

$$i \to j \qquad j \to i$$

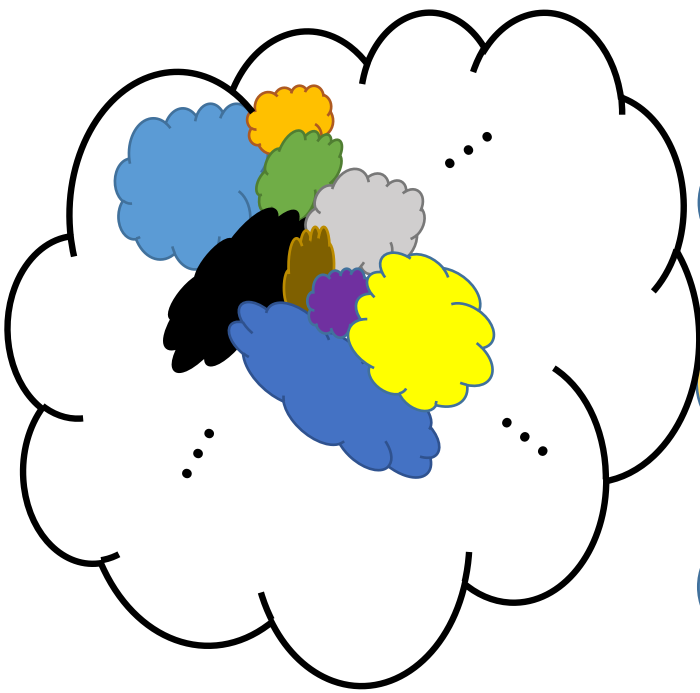
Mass Conservation

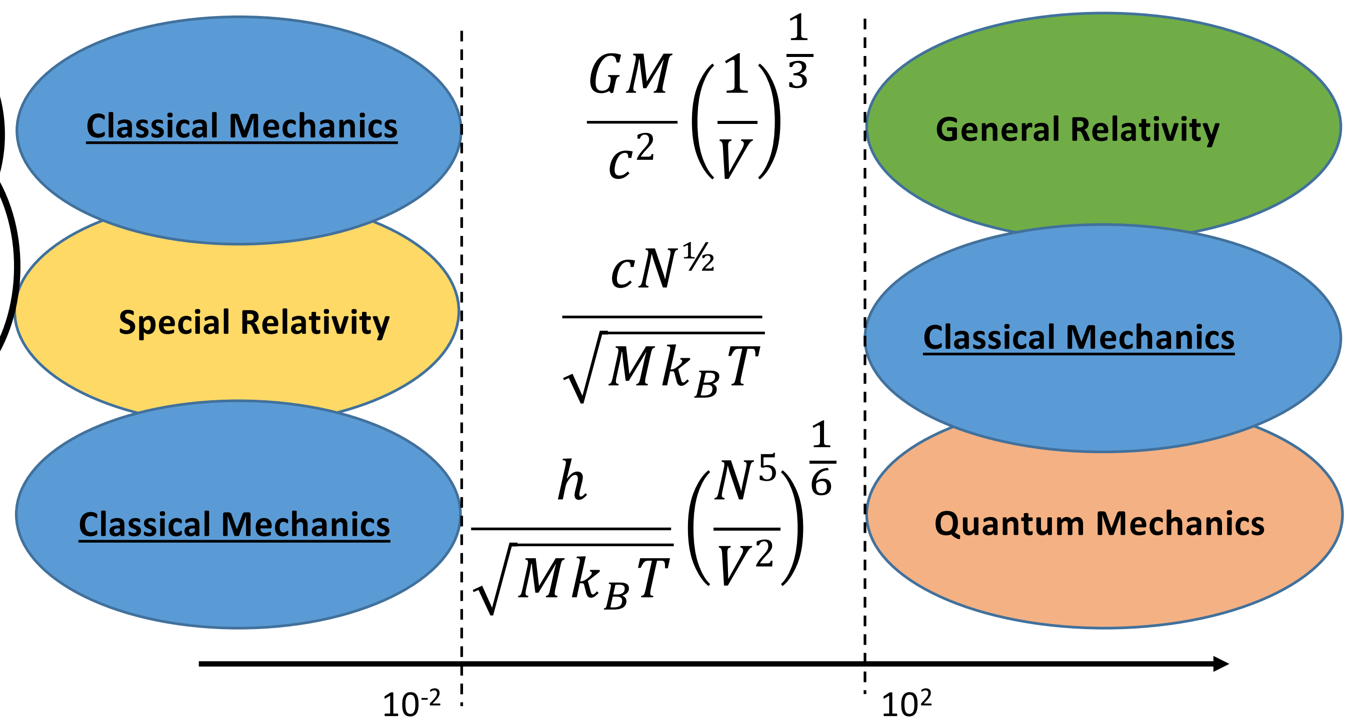


A System of Systems
V", N", ε", {M", E", **P**", **L**"}, ⋯, p", T", **μ**"
S"
V, N, ε, {M, E, **P**, **L**}, ⋯, p, T, **μ**
S
V', N', ε', {M', E', **P**', **L**'}, ⋯, p', T', **μ**'
S'
$\rho(\boldsymbol{x},t)$
$e(\boldsymbol{x},t)$
$\rho(\boldsymbol{x},t)\,\boldsymbol{u}(\boldsymbol{x},t)$
$p(\boldsymbol{x},t)$
$T(\boldsymbol{x},t)$
$\boldsymbol{\mu}(\boldsymbol{x},t)$
$S(\boldsymbol{x},t)$
Dynamic Equilibrium
$p = p' = p'' = \cdots$
$\mathbf{P} = \mathbf{P}' = \mathbf{P}'' = \cdots$
$\mathbf{L} = \mathbf{L}' = \mathbf{L}'' = \cdots$
$e(\boldsymbol{x},t)$
$T(\boldsymbol{x},t)$
$\boldsymbol{\mu}(\boldsymbol{x},t)$
$S(\boldsymbol{x},t)$
Thermal Equilibrium
$T = T' = T'' = \cdots$
$e(\boldsymbol{x},t)$
$\boldsymbol{\mu}(\boldsymbol{x},t)$
$S(\boldsymbol{x},t)$
Chemical Equilibrium
$\boldsymbol{\mu} = \boldsymbol{\mu}' = \boldsymbol{\mu}'' = \cdots$
S is maximal
Thermodynamic Equilibrium

Relaxation Toward Equilibria

… a tale of ***two*** constants: c and h





Note: k_B merely serves to define the temperature scale, and G does the same for *gravitational* mass---and by assumption, *inertial* mass as well.

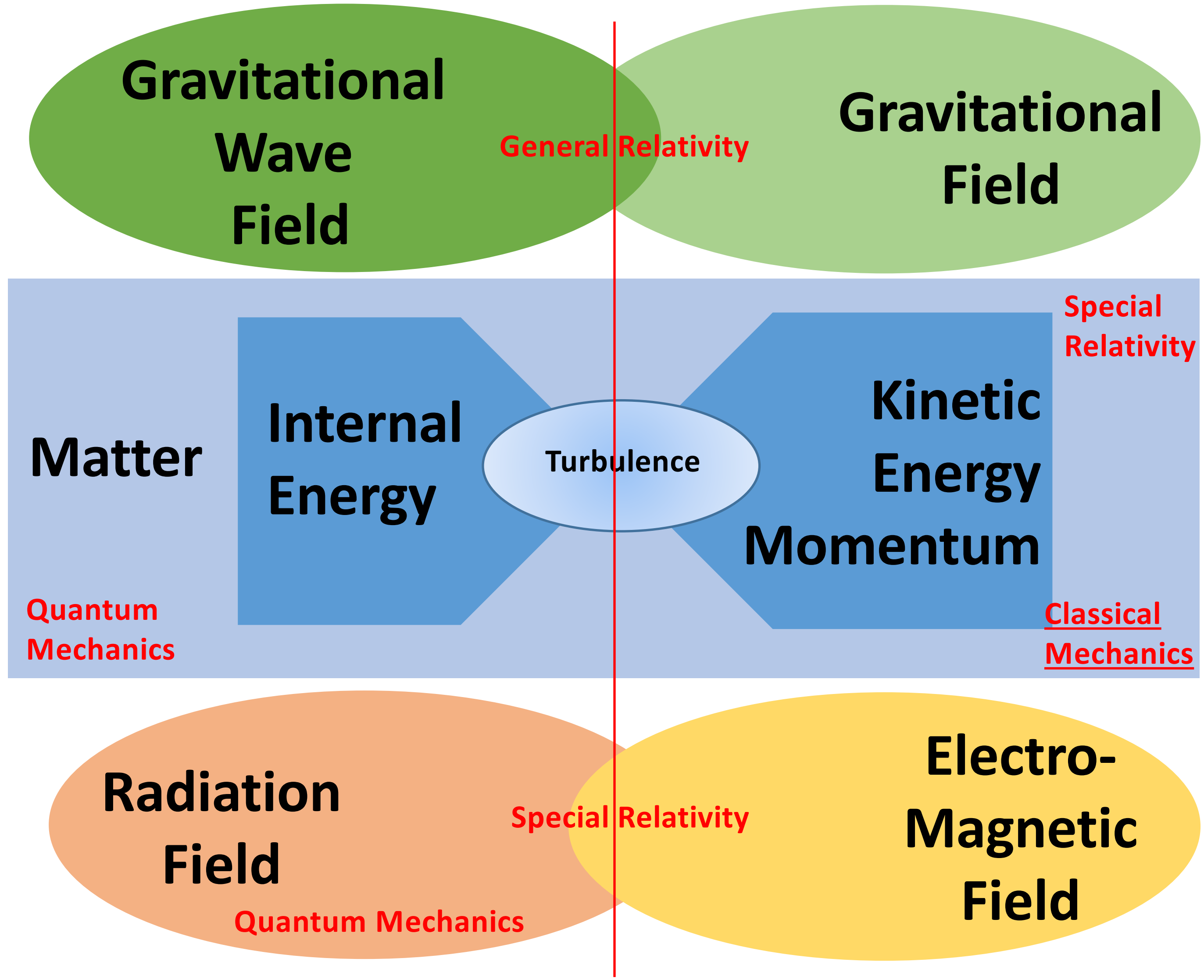
The “*Philosophy of RMHD*”

*“**If** you conserve all the things that need to be conserved **and** you ensure that left to its own devices, entropy always increases, **then** things will often work out far better than one might have any right to expect. (…sometimes)”*

Corollary to the “*Philosophy of RMHD*”

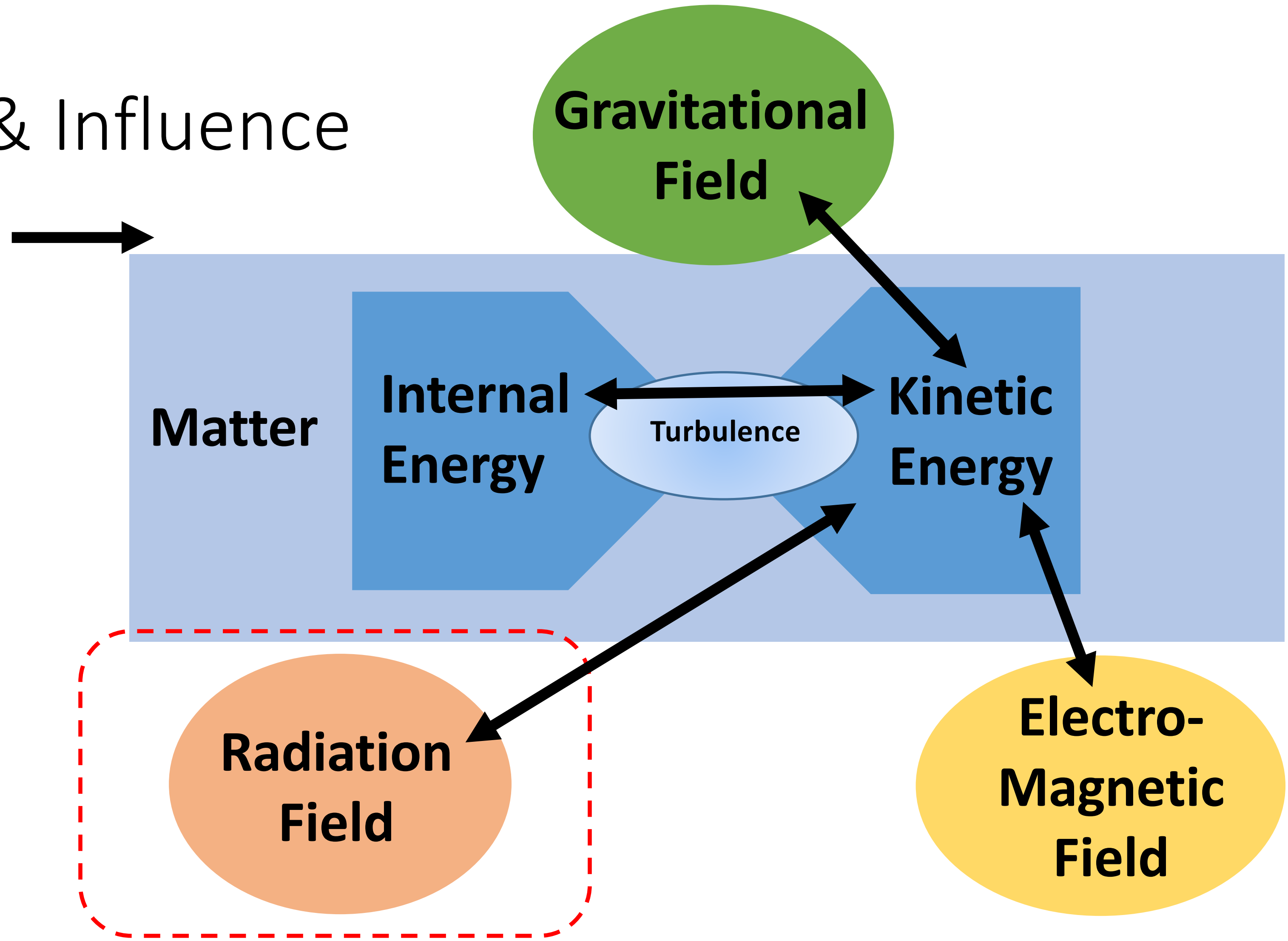
*“**Always** be certain that N is huge, **and** the physical system has both the time and ability to sample lots and lots of its available microstates consistent with a specified macrostate.”*

The Classical Fields: *Statistical* vs *Deterministic*



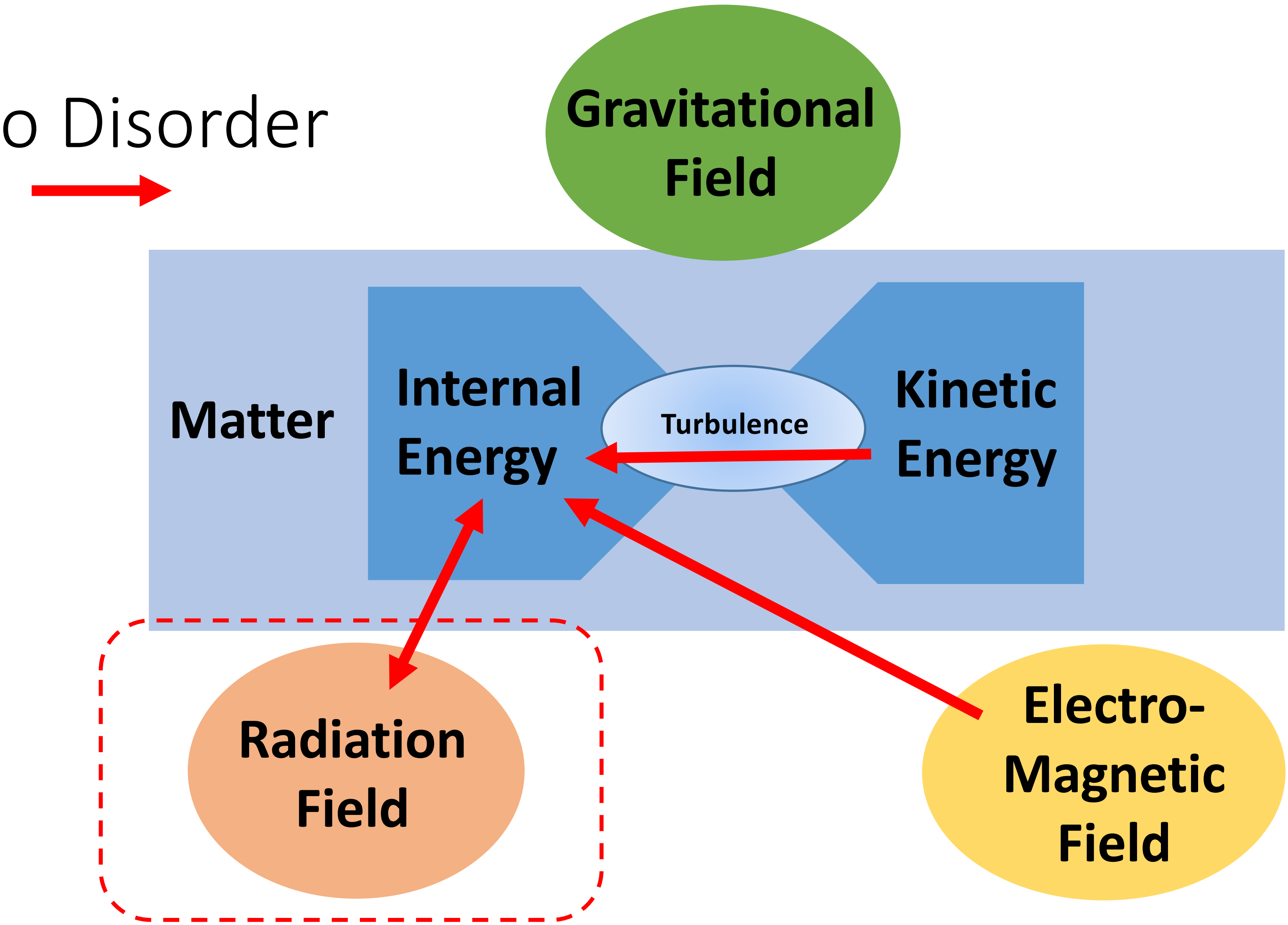
Impacts & Influence

Adiabatic (Reversible)
Entropy is constant

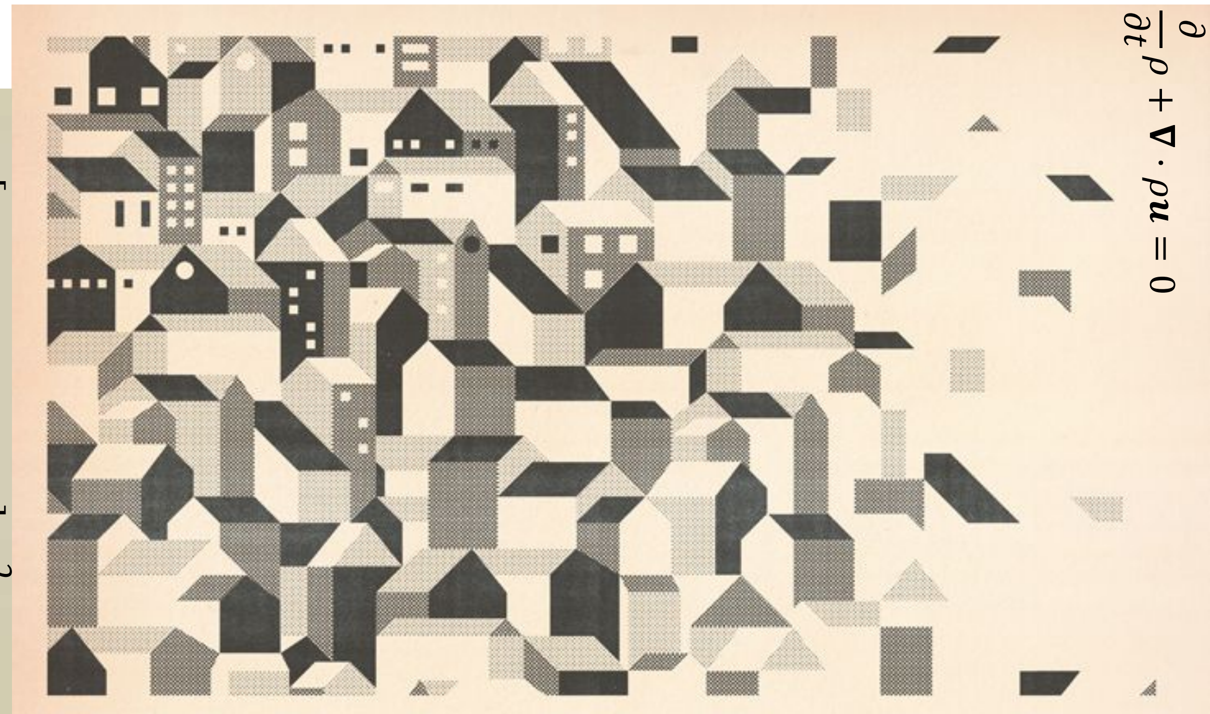


Adding to Disorder

Entropy Production



Construction vs *Deconstruction*



Grâce à Dieu pour Oncle Albert--I

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \frac{8\pi G}{c^4}\left[T_m^{\alpha\beta} + T_{E+M}^{\alpha\beta} + T_{rad}^{\alpha\beta}\right]$$

PHY 3070: RELATIVITÉ 2

PHY 6756: FLUIDES ASTROPHYSIQUE

PHY 3700: ATMOSPHÈRE ET ENVIRONNEMENT STELLAIRES

Grâce à Dieu pour Oncle Albert--II

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \frac{8\pi G}{c^4}\left[T_m^{\alpha\beta} + T_{E+M}^{\alpha\beta} + T_{rad}^{\alpha\beta}\right]$$

$$T_m^{\alpha\beta} = \left(\rho + \frac{p + \rho e}{c^2}\right) U^\alpha U^\beta + p g^{\alpha\beta}$$

$$U^\alpha = \gamma[c, \boldsymbol{u}]^T$$

$$T_{E+M}^{\alpha\beta} = \frac{1}{\mu_0}\left[F^{\alpha\gamma} F_\gamma^{\ \beta} + \frac{1}{2} g^{\alpha\beta}\left(\frac{\|\boldsymbol{E}\|^2}{c^2} - \|\boldsymbol{B}\|^2\right)\right]$$

$$(\nu n)^\alpha = [\nu, \nu \mathbf{n}]^T$$

$$T_{rad}^{\alpha\beta} = \frac{1}{c}\int_0^\infty d\nu \int d\mathbf{n}\, n^\alpha n^\beta I_\nu$$

Full Cost Accounting

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \frac{8\pi G}{c^4}\left[T_m^{\alpha\beta} + T_{E+M}^{\alpha\beta} + T_{rad}^{\alpha\beta}\right]$$

$[0, \mathbf{0}]$

$(\nu n)^{\alpha} = [\nu, \nu\mathbf{n}]^T$

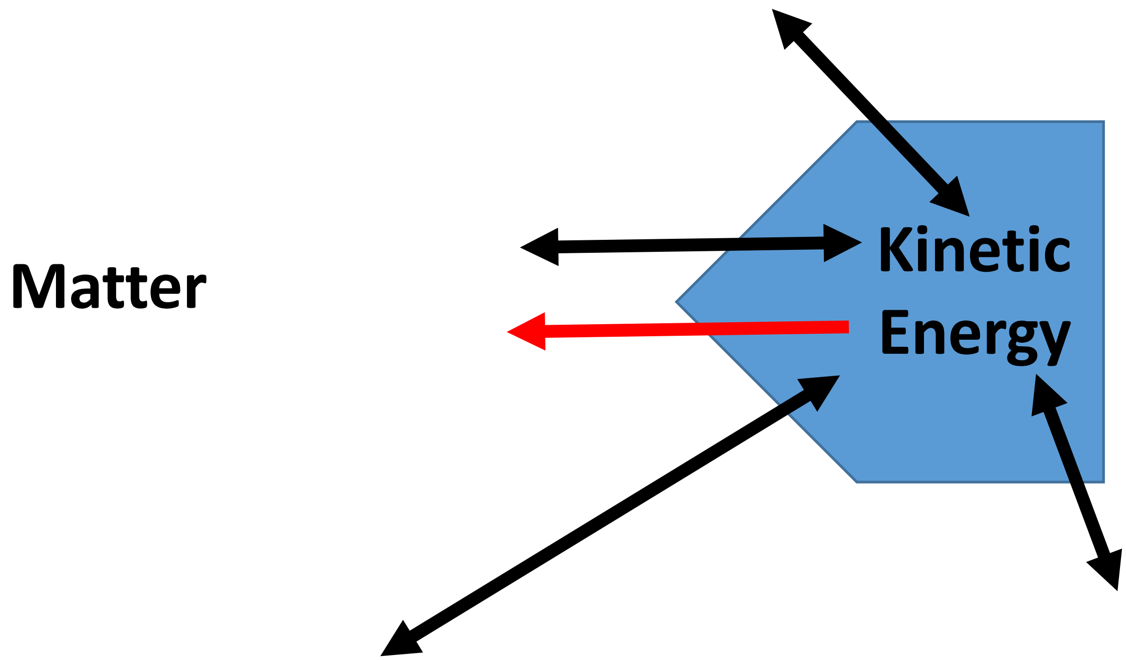
$L^{\alpha} + G^{\alpha}$

$$-L^{\alpha} = [-\boldsymbol{J} \cdot \boldsymbol{E}/c, -\rho_e \boldsymbol{E} - \boldsymbol{J} \times \boldsymbol{B}]^T$$

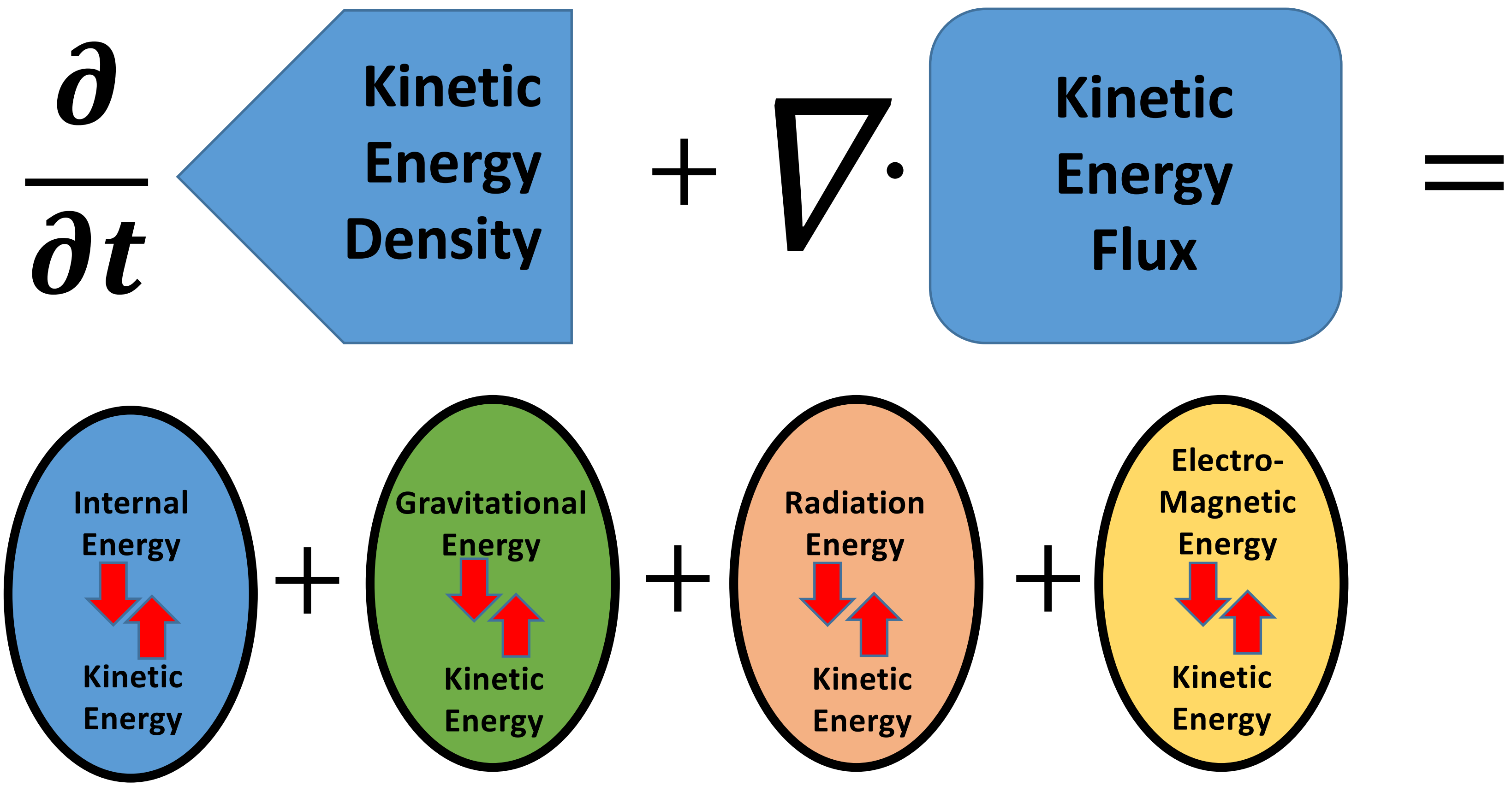
$$-G^{\alpha} = \frac{1}{c}\int_0^{\infty} d\nu \int d\mathbf{n}\, n^{\alpha}[\eta_{\nu} - \chi_{\nu} I_{\nu}]$$

…thank you very much! Any questions?

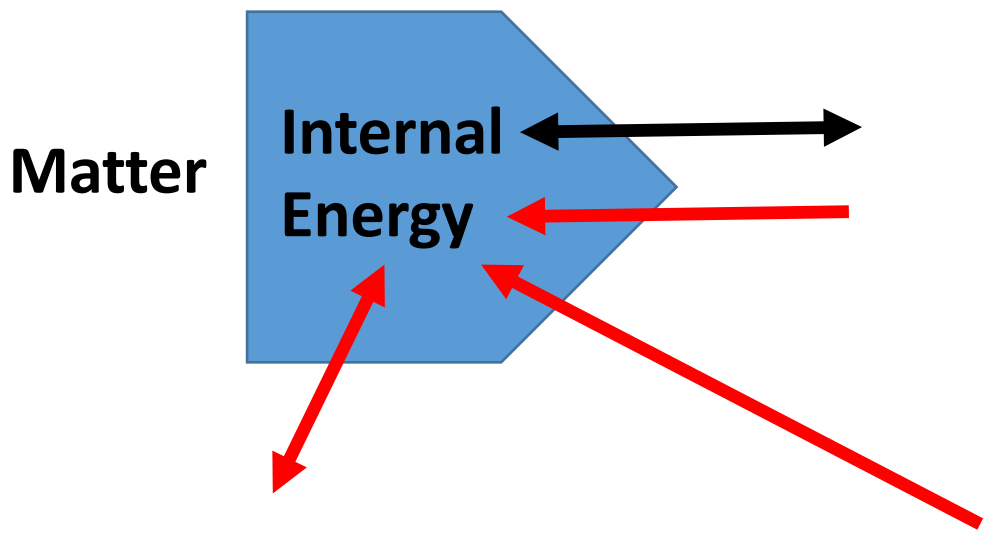
Construction vs Deconstruction



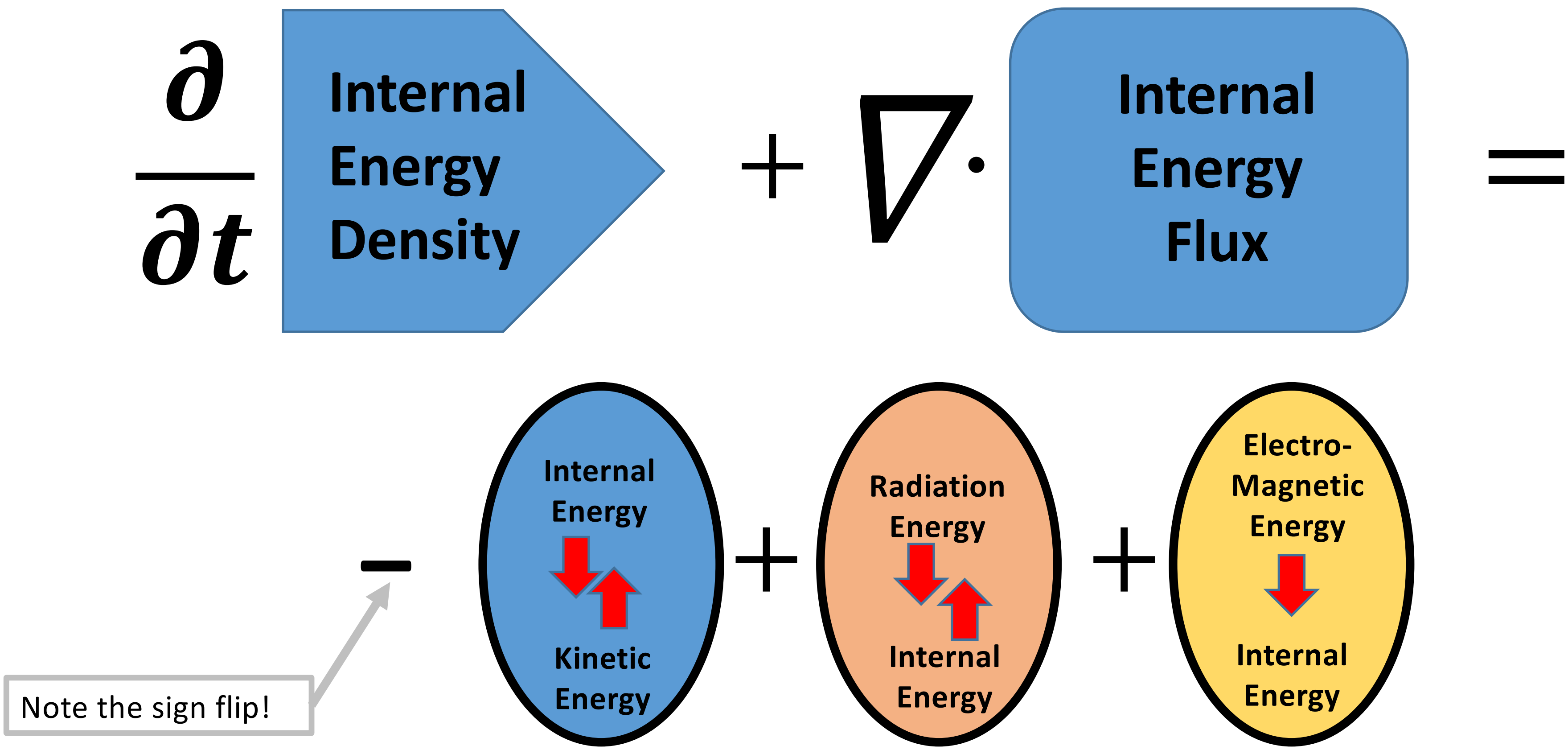
Conservation Laws



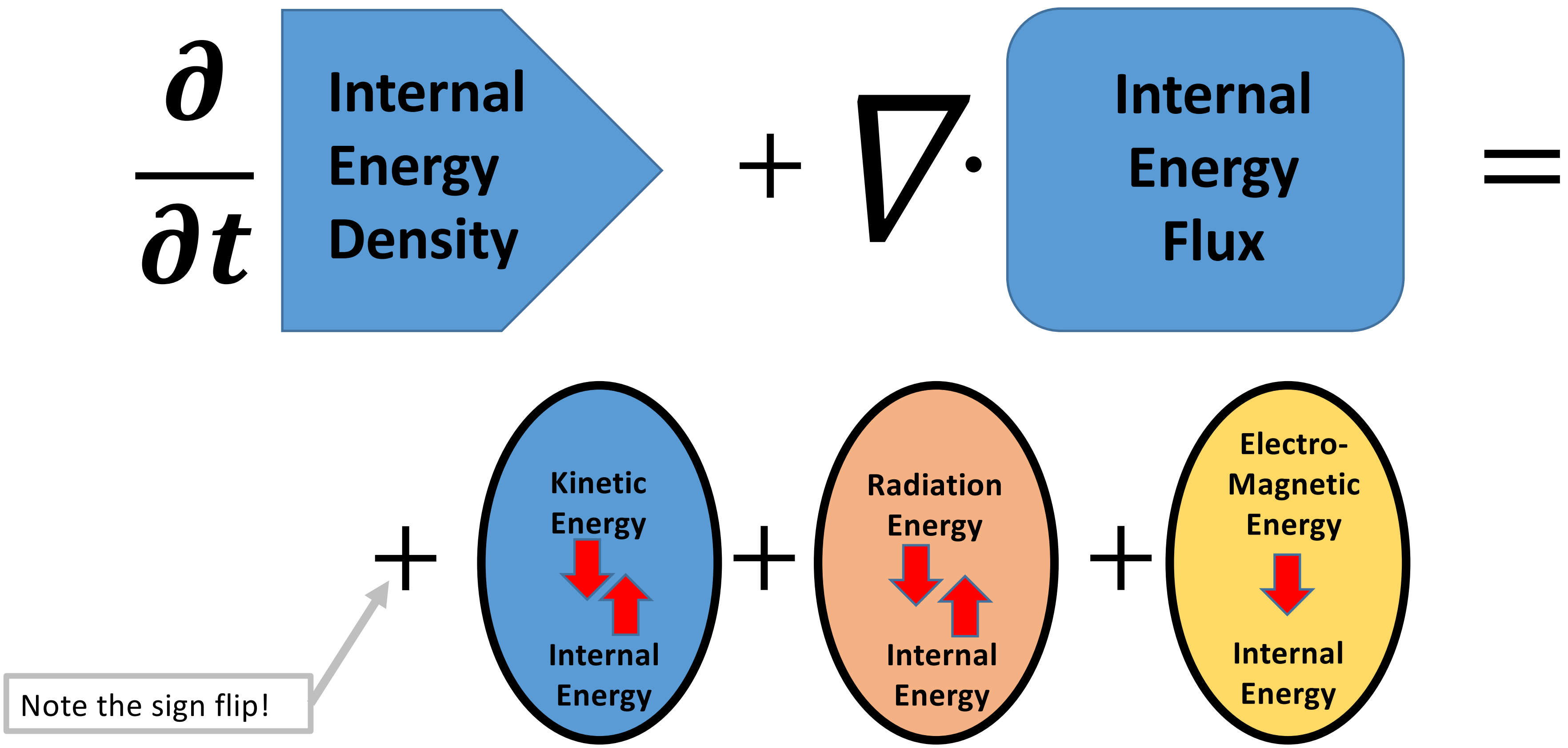
Full Cost Accounting



Conservation Laws



Conservation Laws



The Matter

You have seen this all before!

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\rho\boldsymbol{u} = 0$$

L'équation de continuité (1.21)

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{\nabla}p = -\boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{f}$$

L'équation de Navier-Stokes (1.23)

$$\frac{\partial}{\partial t}e + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = T\dot{s}$$

Conservation de l'énergie interne (1.56)

The Matter

$$\int \mathrm{d}\boldsymbol{p}\left\{ \begin{matrix} 1 \\ \frac{\boldsymbol{p}}{m} \\ \frac{\boldsymbol{p}}{m}\frac{\boldsymbol{p}}{m} \end{matrix} \quad \frac{\partial \psi}{\partial t} + \frac{1}{m}\boldsymbol{p}\cdot\frac{\partial \psi}{\partial \boldsymbol{x}} + \boldsymbol{f}\cdot\frac{\partial \psi}{\partial \boldsymbol{p}} = \cancel{\frac{\delta \psi}{\delta t}} \right\}$$

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\rho\boldsymbol{u} = 0$$

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{\nabla}p = -\boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{f}$$

Trace

$$\frac{\partial}{\partial t}e + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = T\dot{s}$$

We will need to determine these two exchange terms.

The Matter

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \rho \boldsymbol{u} = 0$$

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \frac{1}{\rho}\boldsymbol{f}$$

$$\frac{\partial}{\partial t}e + \boldsymbol{u} \cdot \nabla e + \frac{p}{\rho}\nabla \cdot \boldsymbol{u} = T\dot{s}$$

Note: Thermal conduction and viscous stresses can also be accommodated in $\boldsymbol{f}$ and $T\dot{s}$ if necessary.

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\|\boldsymbol{u}\|^2 + \nabla \cdot \frac{1}{2}\rho\|\boldsymbol{u}\|^2\,\boldsymbol{u} = -\boldsymbol{u} \cdot \nabla p - \boldsymbol{u} \cdot \rho\nabla\Phi + \boldsymbol{u} \cdot \boldsymbol{f}$$

$$\frac{\partial}{\partial t}\rho e + \nabla \cdot (\rho e + p)\,\boldsymbol{u} = +\boldsymbol{u} \cdot \nabla p + \rho T\dot{s}$$

$$\frac{\partial}{\partial t}\rho\boldsymbol{u} + \nabla \cdot (p + \rho\boldsymbol{u}\boldsymbol{u}) = -\rho\,\nabla\Phi + \boldsymbol{f}$$

The Matter

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\rho\boldsymbol{u} = 0$$

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{f}$$

$$\frac{\partial}{\partial t}e + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = T\dot{s}$$

Note: Thermal conduction and viscous stresses can also be accommodated in these terms if desired.

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\|\boldsymbol{u}\|^2 + \boldsymbol{\nabla}\cdot\frac{1}{2}\rho\|\boldsymbol{u}\|^2\,\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p - \boldsymbol{u}\cdot\rho\boldsymbol{\nabla}\Phi + \boldsymbol{u}\cdot\boldsymbol{f}$$

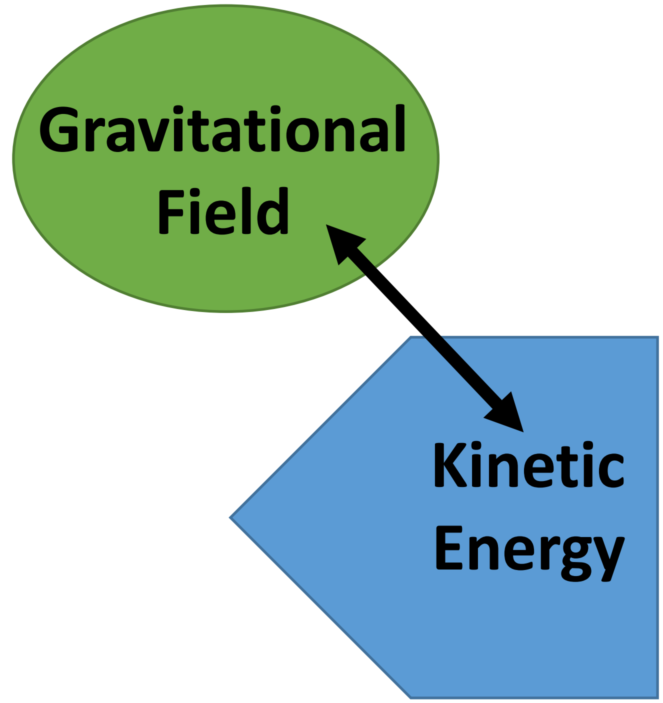
$$\frac{\partial}{\partial t}\rho e + \boldsymbol{\nabla}\cdot(\rho e + p)\,\boldsymbol{u} = +\boldsymbol{u}\cdot\boldsymbol{\nabla}p + \rho T\dot{s}$$

$$\frac{\partial}{\partial t}\rho\boldsymbol{u} + \boldsymbol{\nabla}\cdot(p + \rho\boldsymbol{u}\boldsymbol{u}) = -\rho\,\boldsymbol{\nabla}\Phi + \boldsymbol{f}$$

Someone needs to tell us how to determine the gas pressure!

It remains to determine these terms through full cost accounting!

Full Cost Accounting



The Gravitational Field

You have seen this all before!

$$\nabla^2\Phi = 4\pi G\rho$$

L'équation de Poisson (1.35)

Beware!!! Half the world prefers this way…

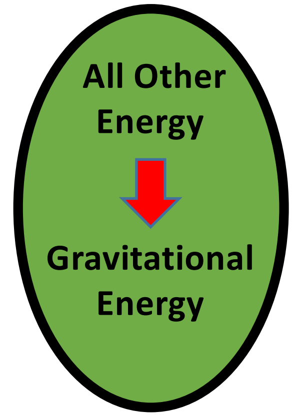
$$\nabla^2\Psi = -4\pi G\rho$$

The Gravitational Field

$$\nabla^2 \Phi = 4\pi G \rho$$

Note: <u>All</u> the gravity is provided by the material we are keeping track of via the continuity equation.

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0$$

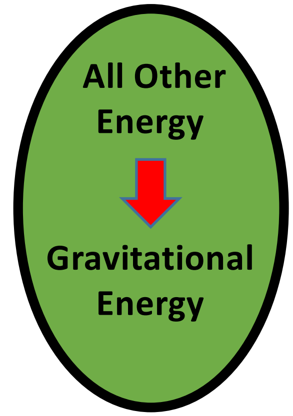


The Gravitational Field

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0$$

Note: All the gravity is provided by the material we are keeping track of via the continuity equation.



Coupling to matter

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\Phi + \boldsymbol{\nabla} \cdot (\rho\Phi\boldsymbol{u} + \boldsymbol{G}) = \rho\boldsymbol{u} \cdot \boldsymbol{\nabla}\Phi$$

$$\boldsymbol{\nabla} \cdot \mathbb{G} = \rho\boldsymbol{\nabla}\Phi$$

Your HOMEWORK Assignment!

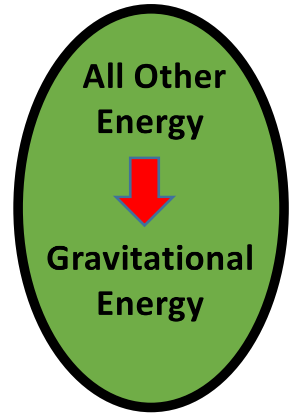
Compute $\boldsymbol{G}$ and $\mathbb{G}$.

The Gravitational Field

$$\nabla^2\Phi = 4\pi G\rho$$

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\rho\boldsymbol{u} = 0$$

Note: None of the gravity is provided by the material we are keeping track of via the continuity equation.

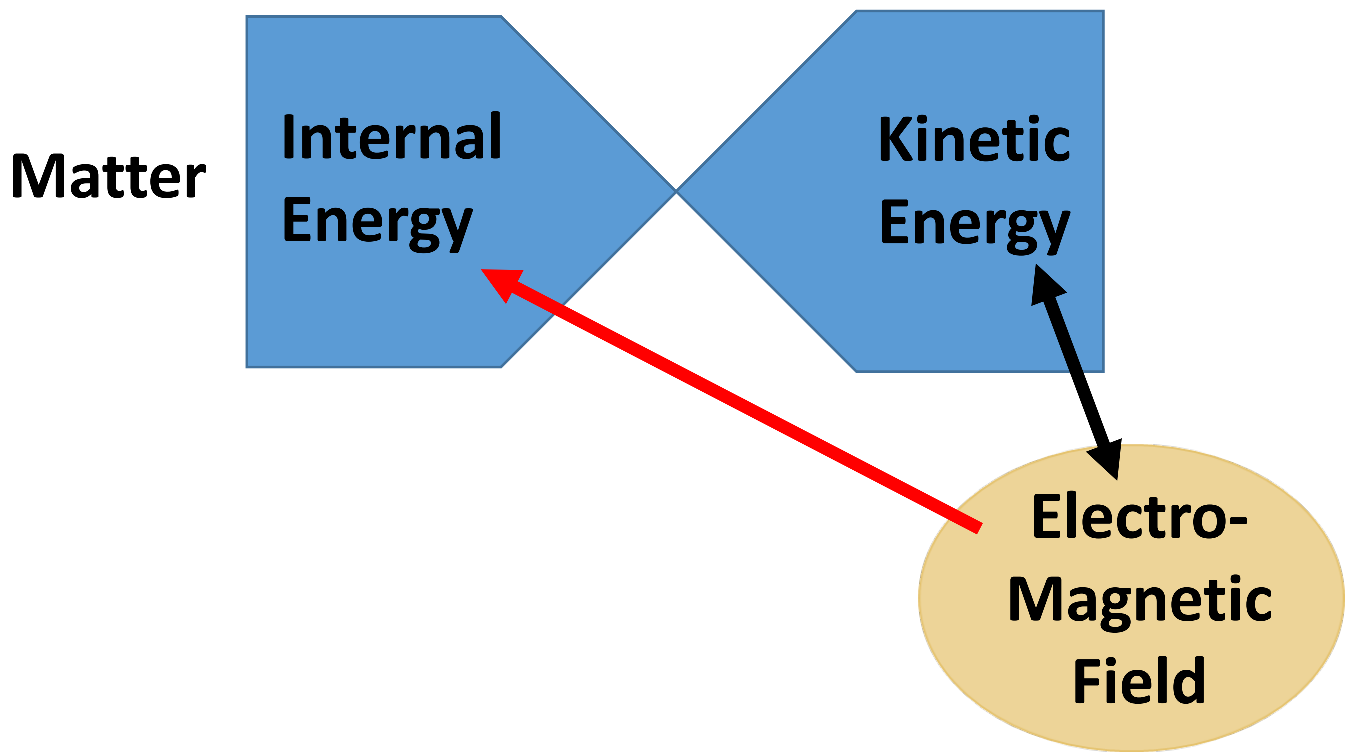


$$\frac{\partial}{\partial t}\frac{1}{2}\rho\Phi + \boldsymbol{\nabla}\cdot(\rho\Phi\boldsymbol{u} + \boldsymbol{G}) = \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi$$

$$\boldsymbol{\nabla}\cdot\mathbb{G} = \rho\boldsymbol{\nabla}\Phi$$

Someone needs to tell us how to determine the gravitational potential Φ!

Full Cost Accounting



The Electromagnetic Field

You have seen this all before!

$$\nabla \cdot \boldsymbol{E} = \rho_e/\varepsilon_0$$

Loi de Gauss (2.1)

$$\nabla \cdot \boldsymbol{B} = 0$$

Loi anonyme (2.2)

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Loi de Faraday (2.3)

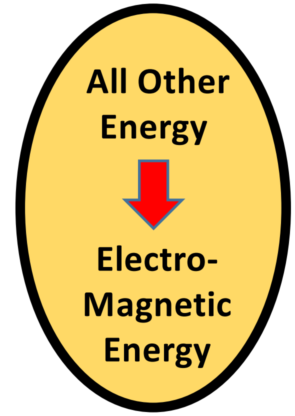
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Loi d'Ampère/Maxwell (2.4)

The Electromagnetic Field

$$\nabla \cdot \boldsymbol{E} = \rho_e/\varepsilon_0 \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$



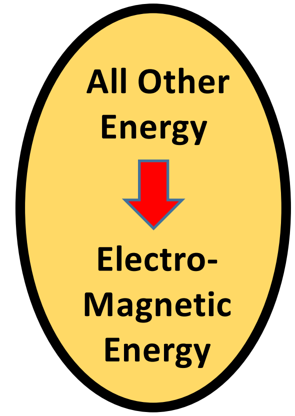
$$\frac{\partial}{\partial t}\frac{1}{2\mu_0}\left(\mu_0\varepsilon_0\|\boldsymbol{E}\|^2+\|\boldsymbol{B}\|^2\right) + \nabla \cdot \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{J}\cdot\boldsymbol{E}$$

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{S}}{\partial t} + \nabla\cdot\mathbb{M} = -\rho_e\boldsymbol{E} - \boldsymbol{J}\times\boldsymbol{B}$$

Someone needs to tell us how to determine the electric current $\boldsymbol{J}$ and charge density ρ_e !

Coupling to matter

The Electromagnetic Field



$$\nabla \cdot \boldsymbol{E} = \rho_e/\varepsilon_0 \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

MHD scalings: $u^3B^2/\ell c^2$ uB^2/ℓ uB^2/ℓ uB^2/ℓ

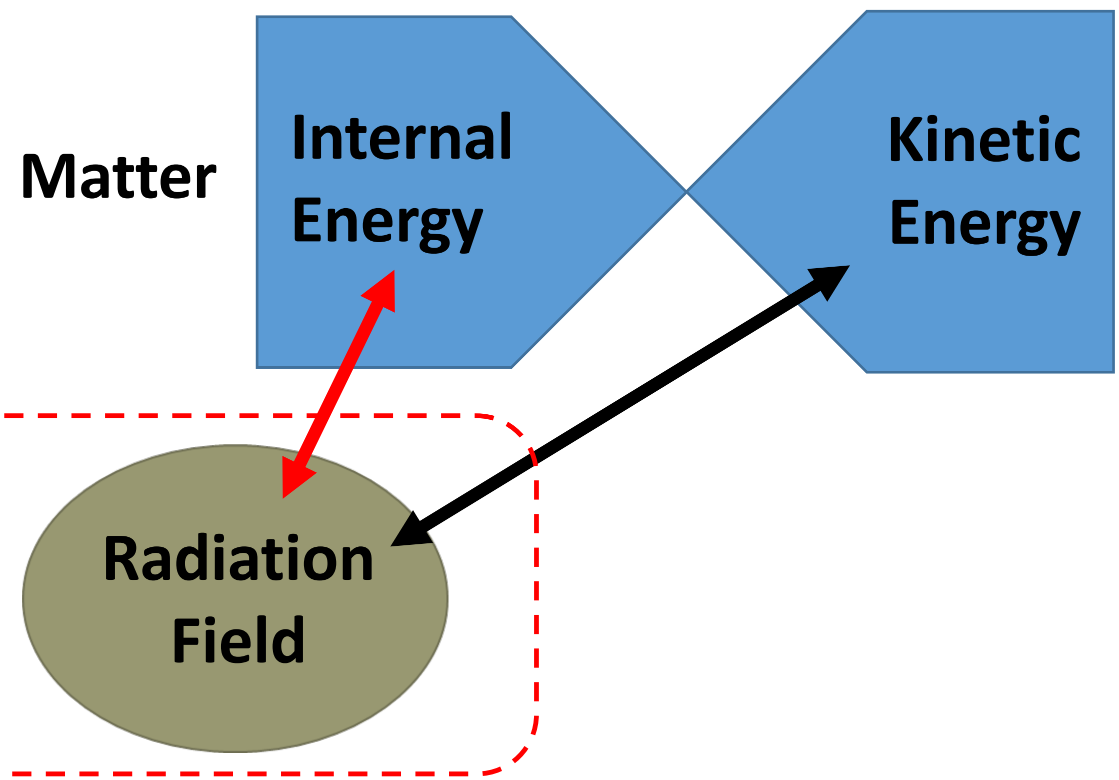
$$\frac{\partial}{\partial t}\frac{1}{2\mu_0}\left(\cancel{\mu_0\varepsilon_0\|\boldsymbol{E}\|^2} + \|\boldsymbol{B}\|^2\right) + \nabla \cdot \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} = -\boldsymbol{J} \cdot \boldsymbol{E}$$

$$\cancel{\frac{1}{c^2} \cdot \frac{\partial \boldsymbol{S}}{\partial t}} + \nabla \cdot \mathbb{M} = \cancel{-\rho_e \boldsymbol{E}} - \boldsymbol{J} \times \boldsymbol{B}$$

Your HOMEWORK Assignment!

Compute $\mathbb{M}$.

Full Cost Accounting



The Radiation Field

You have seen this all before!

Conservation de l'énergie interne (1.56)

$$\frac{\partial}{\partial t} e + \boldsymbol{u} \cdot \boldsymbol{\nabla} e + \frac{p}{\rho} \boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{1}{\rho} \boldsymbol{\nabla} \cdot [(\chi + \chi_r) \boldsymbol{\nabla} T]$$

$$\frac{\partial}{\partial t} S + \boldsymbol{u} \cdot \boldsymbol{\nabla} S = \frac{1}{\rho T} \boldsymbol{\nabla} \cdot [(\chi + \chi_r) \boldsymbol{\nabla} T]$$

Deuxième principe de la thermodynamique (1.58)

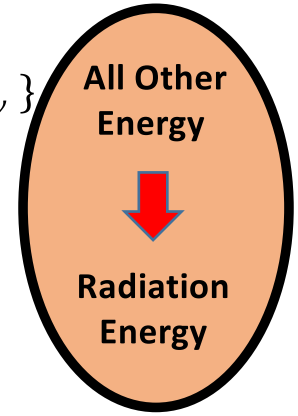
$\dot{S}$

The Radiation Field

$$\frac{1}{c}\cdot\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = \eta_\nu - \chi_\nu I_\nu$$

Note: Extension to polarized radiative transfer is possible.

$$\oint \mathrm{d}\boldsymbol{n}\int \mathrm{d}\nu \left[c^{-1},\ \frac{\mathbf{n}}{c},\ \frac{\mathbf{n}}{c}\frac{\mathbf{n}}{c}\right]\left\{\frac{\partial I_\nu}{\partial t} + c\mathbf{n}\cdot\frac{\partial I_\nu}{\partial \boldsymbol{x}} = c\eta_\nu - c\chi_\nu I_\nu\right\}$$



Coupling to matter

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = \int_0^\infty d\nu\int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

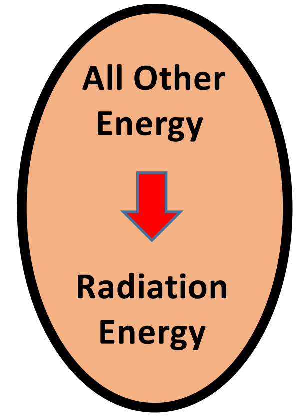
$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + \nabla\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu\int d\mathbf{n}\,\mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$

Someone needs to tell us how to determine the radiation pressure tensor $\mathbb{P}$, emissivity η_ν , and opacity χ_ν!

The Radiation Field

$$\frac{1}{c}\cdot\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = \eta_\nu - \chi_\nu I_\nu$$

Note: Extension to polarized radiative transfer is possible.



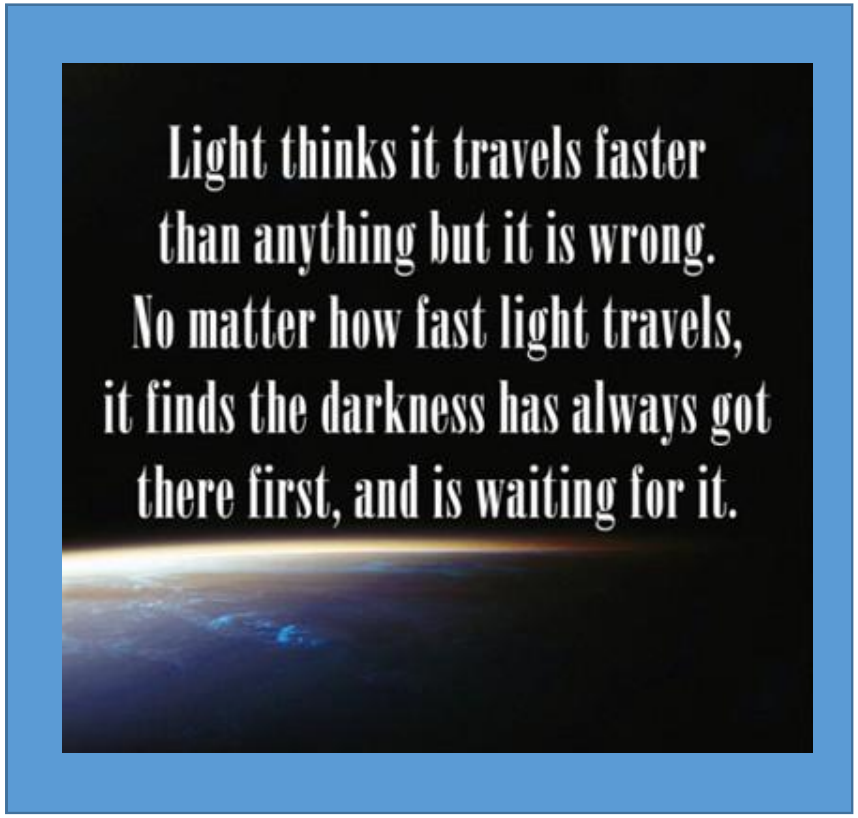
*Optically **thick** to thin scalings:*

uE/ℓ *(**u** to c)E/ℓ* *?* *cE/λ*

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = \int_0^\infty d\nu \int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

It is the rare occasion when one actually needs to retain this term!

$$\cancel{\frac{1}{c^2}\frac{\partial \boldsymbol{F}}{\partial t}} + \nabla\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu \int d\mathbf{n}\,\mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$


Light thinks it travels faster
than anything but it is wrong.
No matter how fast light travels,
it finds the darkness has always got
there first, and is waiting for it.

Summary

Momentum Conservation

$$\frac{\partial}{\partial t}\rho\boldsymbol{u} + \boldsymbol{\nabla}\cdot(p + \rho\boldsymbol{u}\boldsymbol{u}) = -\rho\,\boldsymbol{\nabla}\Phi + \boldsymbol{f}$$

$$\boldsymbol{\nabla}\cdot\mathbb{G} = \rho\boldsymbol{\nabla}\Phi$$

$$\frac{1}{c^2}\cdot\frac{\partial\boldsymbol{S}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{M} = -\rho_e\boldsymbol{E} - \boldsymbol{J}\times\boldsymbol{B}$$

$$\frac{1}{c^2}\cdot\frac{\partial\boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu\int d\mathbf{n}\,\mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$

$$\frac{\partial\mathfrak{P}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{II} = 0$$

Momentum Conservation

$$\frac{\partial}{\partial t}\rho \boldsymbol{u} + \boldsymbol{\nabla}\cdot(p + \rho \boldsymbol{u}\boldsymbol{u}) = -\rho\,\boldsymbol{\nabla}\Phi + \boldsymbol{f}$$

$$\boldsymbol{\nabla}\cdot\mathbb{G} = \rho\boldsymbol{\nabla}\Phi$$

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{S}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{M} = -\rho_e\,\boldsymbol{E} - \boldsymbol{J}\times\boldsymbol{B}$$

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu \int d\mathbf{n}\,\mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$

$$\frac{\partial \mathfrak{P}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{\Pi} = 0$$

Summary

Momentum Conservation

Now we can determine the force density!

$$\frac{\partial}{\partial t}\rho\boldsymbol{u} + \boldsymbol{\nabla}\cdot(p + \rho\boldsymbol{u}\boldsymbol{u}) = -\rho\,\boldsymbol{\nabla}\Phi + \boldsymbol{f}$$

$$\boldsymbol{\nabla}\cdot\mathbb{G} = \rho\boldsymbol{\nabla}\Phi$$

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{S}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{M} = -\rho_e\,\boldsymbol{E} - \boldsymbol{J}\times\boldsymbol{B}$$

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu\int d\mathbf{n}\,\mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$

$$\frac{\partial \mathfrak{P}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{\Pi} = 0$$

Summary

Energy Conservation

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\|\boldsymbol{u}\|^2 + \boldsymbol{\nabla}\cdot\frac{1}{2}\rho\|\boldsymbol{u}\|^2\,\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi + \boldsymbol{u}\cdot\boldsymbol{f}$$

$$\frac{\partial}{\partial t}\rho e + \boldsymbol{\nabla}\cdot(\rho e + p)\boldsymbol{u} = +\boldsymbol{u}\cdot\boldsymbol{\nabla}p + \rho T\dot{s}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\Phi + \boldsymbol{\nabla}\cdot(\rho\Phi\boldsymbol{u} + \boldsymbol{G}) = \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi$$

$$\frac{\partial}{\partial t}\frac{1}{2\mu_0}(\mu_0\varepsilon_0\|\boldsymbol{E}\|^2+\|\boldsymbol{B}\|^2) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{J}\cdot\boldsymbol{E}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = \int_0^\infty d\nu\int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

$$\frac{\partial\mathfrak{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{F}} = 0$$

Summary

Energy Conservation

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\|\boldsymbol{u}\|^2 + \boldsymbol{\nabla}\cdot\frac{1}{2}\rho\|\boldsymbol{u}\|^2\,\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi + \boldsymbol{u}\cdot\boldsymbol{f}$$

$$\frac{\partial}{\partial t}\rho e + \boldsymbol{\nabla}\cdot(\rho e + p)\boldsymbol{u} = +\boldsymbol{u}\cdot\boldsymbol{\nabla}p + \rho T\dot{s}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\Phi + \boldsymbol{\nabla}\cdot(\rho\Phi\boldsymbol{u} + \boldsymbol{G}) = \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi$$

$$\frac{\partial}{\partial t}\frac{1}{2\mu_0}(\mu_0\varepsilon_0\|\boldsymbol{E}\|^2+\|\boldsymbol{B}\|^2) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{J}\cdot\boldsymbol{E}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = \int_0^\infty d\nu\int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

$$\frac{\partial\mathfrak{E}}{\partial t} + \boldsymbol{\nabla}\cdot\mathfrak{F} = 0$$

Summary

Energy Conservation

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\|\boldsymbol{u}\|^2 + \boldsymbol{\nabla}\cdot\frac{1}{2}\rho\|\boldsymbol{u}\|^2\,\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi + \boldsymbol{u}\cdot\boldsymbol{f}$$

$$\frac{\partial}{\partial t}\rho e + \boldsymbol{\nabla}\cdot(\rho e + p)\boldsymbol{u} = +\boldsymbol{u}\cdot\boldsymbol{\nabla}p + \rho T\dot{s}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\rho\Phi + \boldsymbol{\nabla}\cdot(\rho\Phi\boldsymbol{u} + \boldsymbol{G}) = \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\Phi$$

$$\frac{\partial}{\partial t}\frac{1}{2\mu_0}(\mu_0\varepsilon_0\|\boldsymbol{E}\|^2 + \|\boldsymbol{B}\|^2) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{J}\cdot\boldsymbol{E}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = \int_0^\infty d\nu\int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

$$\frac{\partial\mathfrak{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{F}} = 0$$

Now we can determine the heat addition to the material!

Summary

Full Cost Accounting---*Done!*

$$\boldsymbol{f} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - \frac{1}{c}\int_0^\infty d\nu \int d\mathbf{n}\, \mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$

$$\rho T \dot{s} = \boldsymbol{J} \cdot \boldsymbol{E} - \boldsymbol{u} \cdot \boldsymbol{f} - \int_0^\infty d\nu \int d\mathbf{n}\, [\eta_\nu - \chi_\nu I_\nu]$$

It remains to determine these terms through the geometry of space-time!

This slide is the ***essential*** objective of RMHD---we have now constructed a set of equations that ***not only*** conserve total energy and momentum, ***but also*** describe how energy and momentum are exchanged between **matter** and the **radiation**, **gravitational** and **electromagnetic** fields!

The "*Golden Rule of RMHD*"

"***Always*** *evaluate interactions between the matter and the classical fields in the* ***comoving****, e.g., rest-frame, of the material!!!"*

but…

*"****Solve*** *your equations in whatever is the most* ***convenient*** *frame of reference for your objectives."*

Can I get an *AMEN, people!!!*

Corollary to the “*Golden Rule of RMHD*”

*“You had better know how to transform coordinates, physical quantities, fields, differential and integral operators (and anything else you can think of) between **any** two frames of reference, under **all** conditions.”*

Abandon hope, all ye who fail to heed the Corollary!

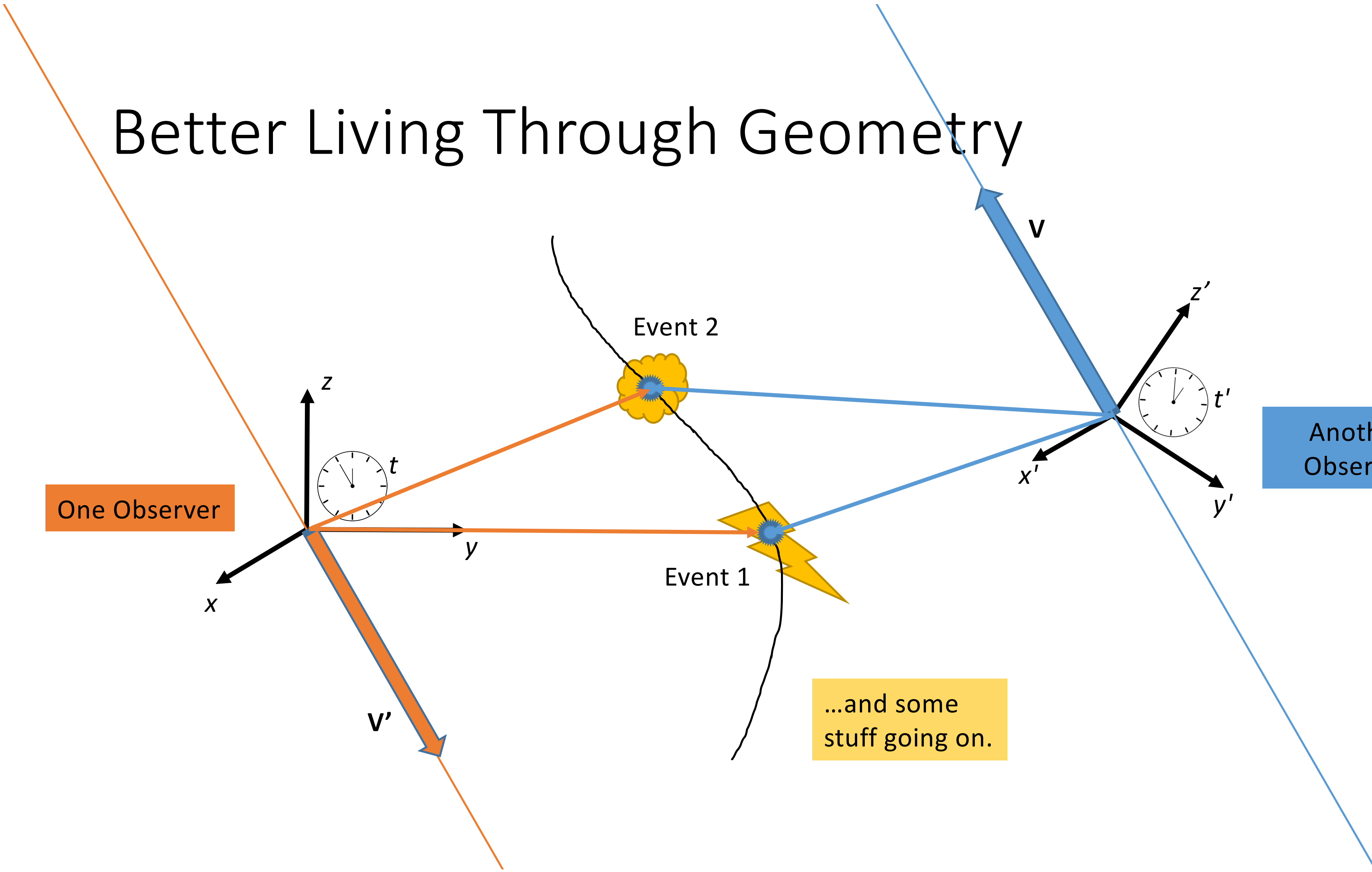
Astrophysical Radiation Magnetohydrodynamics

Being an Opera in Four Acts

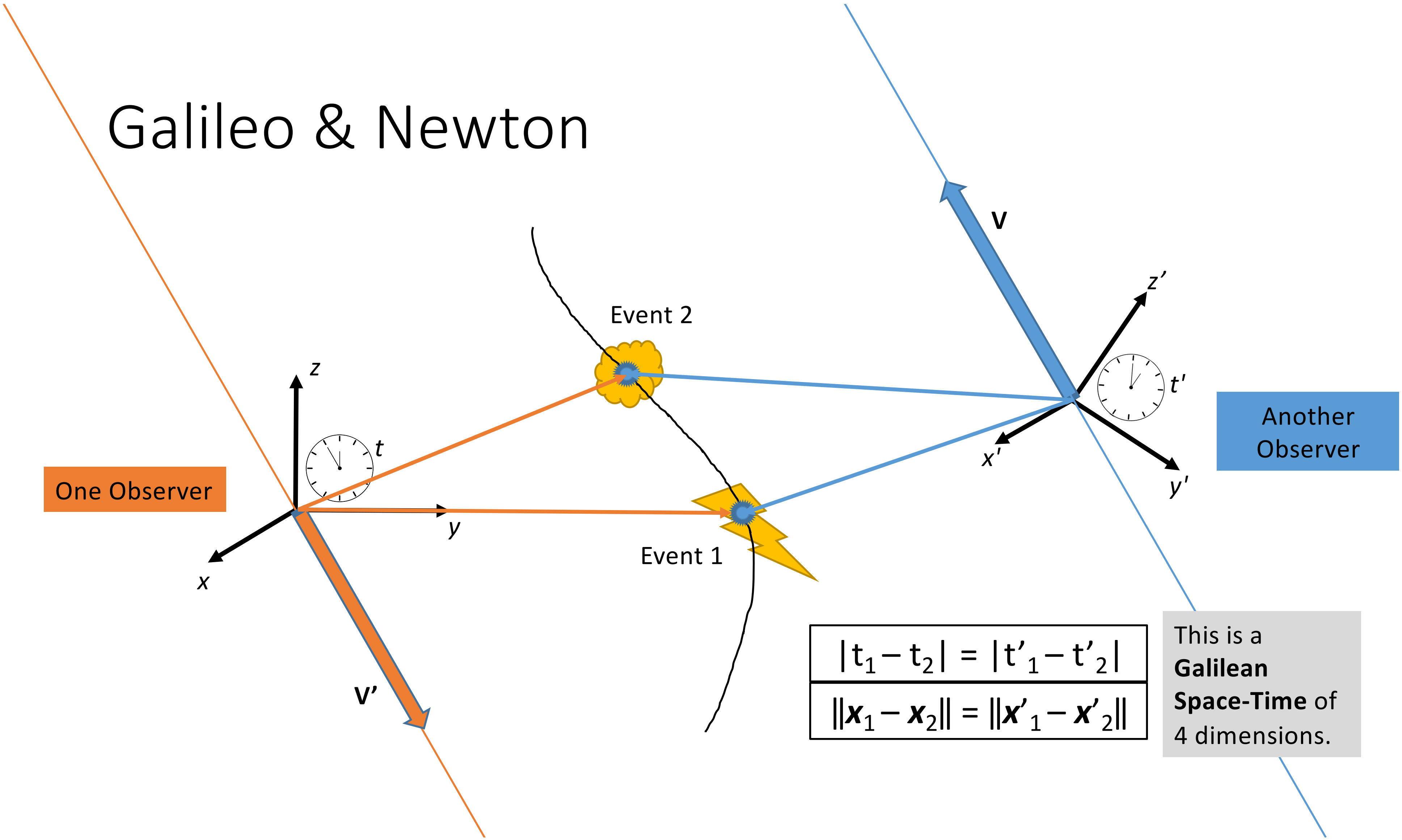
ACT II

Now is the geometry of our discontent,
Made gloriously invariant by this Sun of France;
And all the aether that lowered upon our house,
In the deep bosom of obscurity is buried.

Better Living Through Geometry



Galileo & Newton

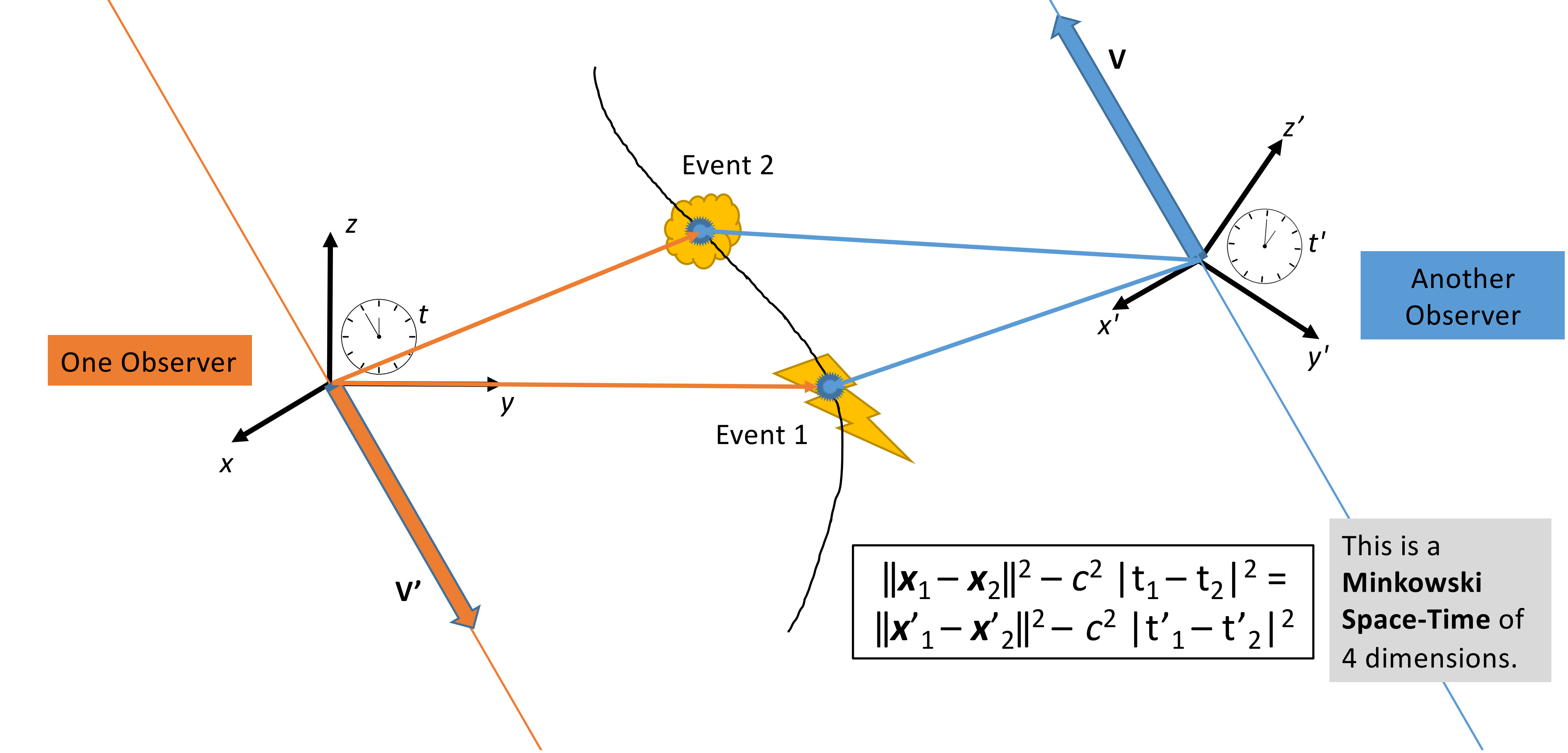


$$|t_1 - t_2| = |t'_1 - t'_2|$$

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| = \|\boldsymbol{x}'_1 - \boldsymbol{x}'_2\|$$

This is a **Galilean Space-Time** of 4 dimensions.

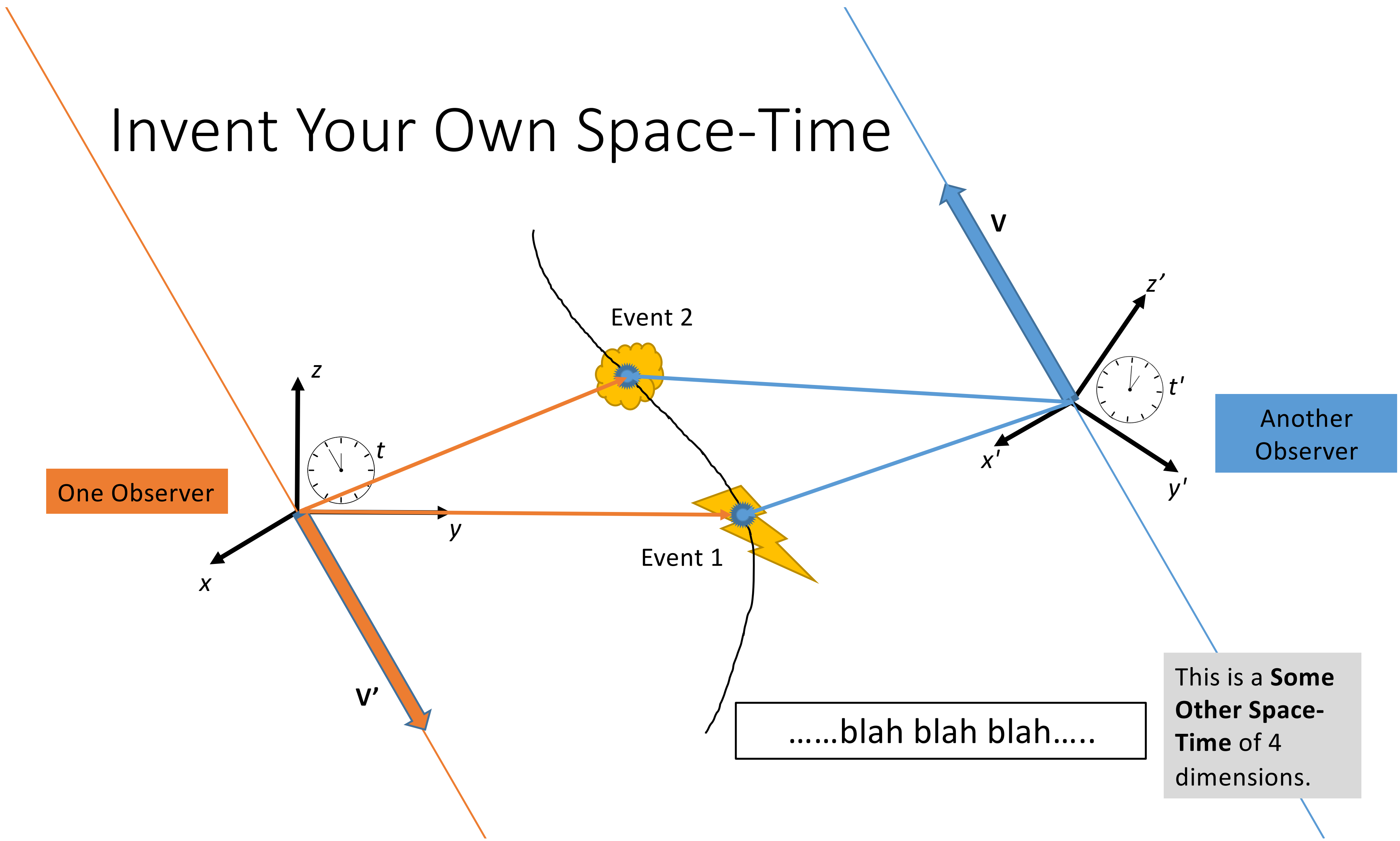
Poincaré, Lorentz & Einstein



$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2 - c^2\ |\mathrm{t}_1 - \mathrm{t}_2|^2 = \|\boldsymbol{x}'_1 - \boldsymbol{x}'_2\|^2 - c^2\ |\mathrm{t}'_1 - \mathrm{t}'_2|^2$$

This is a **Minkowski Space-Time** of 4 dimensions.

Invent Your Own Space-Time



Count the Invariants

Minkowski Space-Time

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2 - c^2\,|t_1 - t_2|^2 = \|\boldsymbol{x}'_1 - \boldsymbol{x}'_2\|^2 - c^2\,|t'_1 - t'_2|^2$$

Galilean Space-Time

$$|t_1 - t_2| = |t'_1 - t'_2|$$

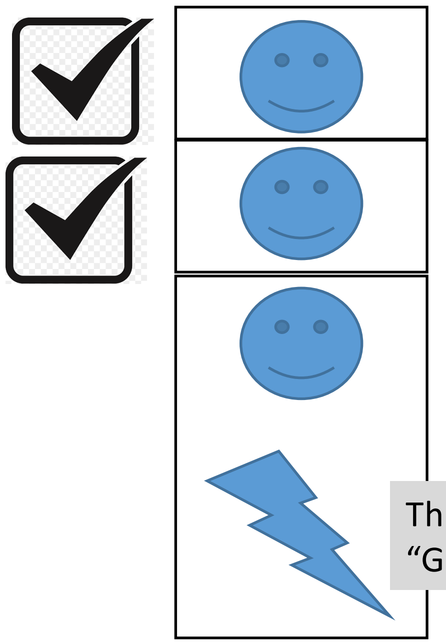
$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| = \|\boldsymbol{x}'_1 - \boldsymbol{x}'_2\|$$

1 Pick a different origin for marking off ***time***.

3 Pick a different origin for marking off ***space***.

3 Pick a different ***orientation*** for your coordinate axes.

3 ***Move*** through the space at a constant rectilinear velocity.

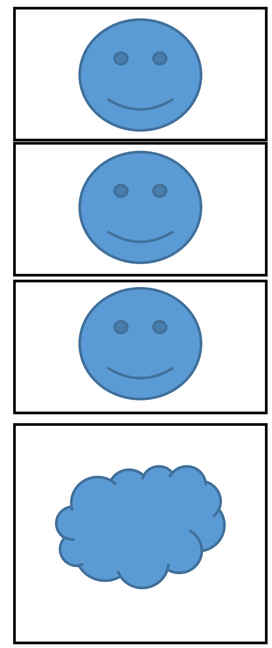


The Translation Group T(1)

The Translation Group T(3)

The Special Orthogonal Group SO(3)

The Lorentz "Group"



Translations

Translations

Proper Rotations

Boosts

10 Parameters

The Poincaré Group

The Galilean Group

Group Action on 4-Vectors

$$\begin{bmatrix} ct' \\ \boldsymbol{x}' \end{bmatrix} \xrightarrow[\mathcal{G}]{\mathcal{L}} \begin{bmatrix} ct'' \\ \boldsymbol{x}'' \end{bmatrix} \xrightarrow[\mathcal{G}^{-1}]{\mathcal{L}^{-1}} \begin{bmatrix} ct' \\ \boldsymbol{x}' \end{bmatrix}$$

Lorentz Group

Galilean Group

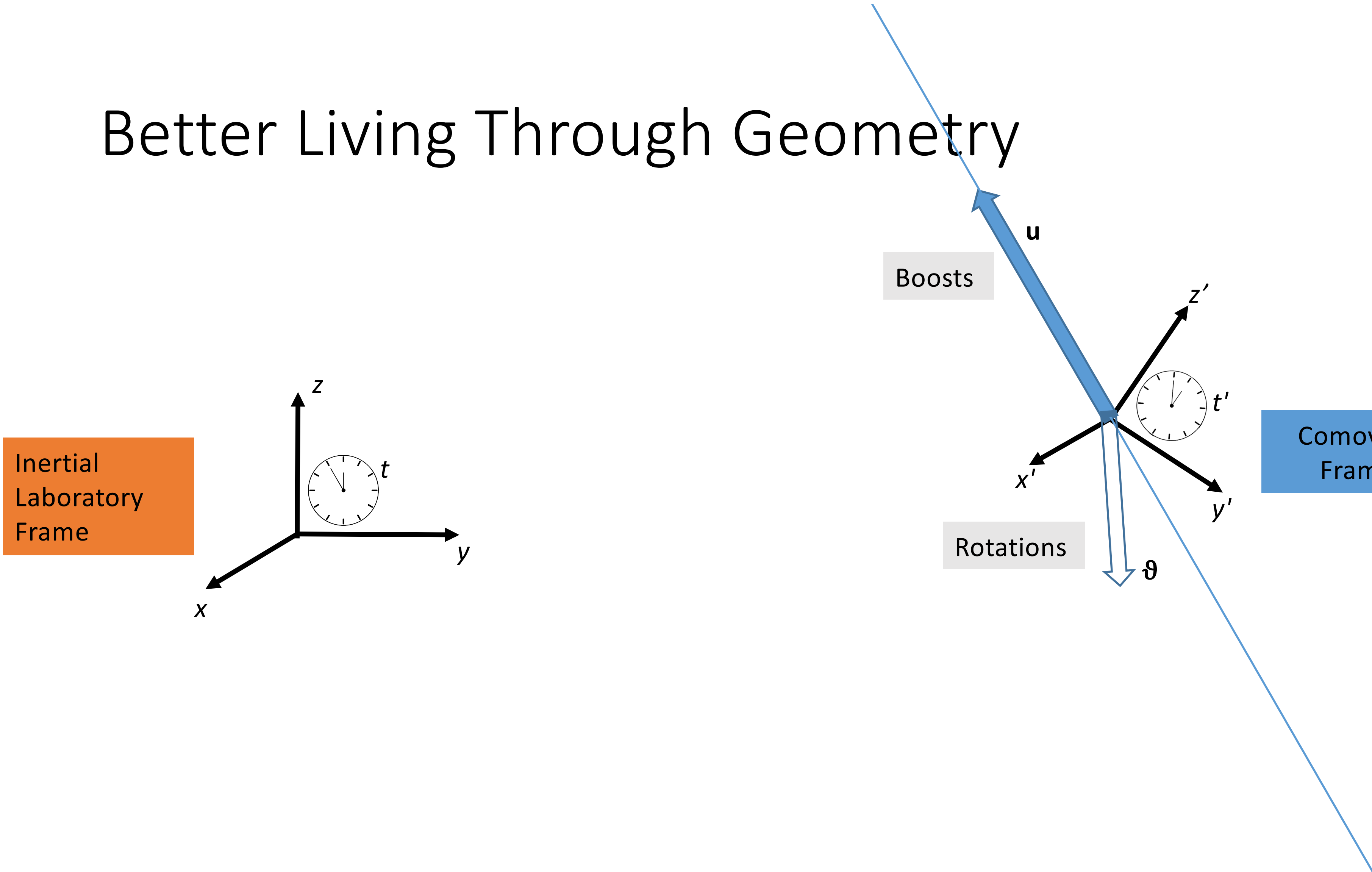
$$\begin{bmatrix} c\rho_e{}' \\ \boldsymbol{J}' \end{bmatrix} \quad \begin{bmatrix} \nu' \\ \nu'\mathbf{n}' \end{bmatrix}$$

Bonus: These 4-vectors live in the *tangent space* to each point in the space-time, and they transform in the same fashion as the space-time itself!

The electric and magnetic fields are components of a 4-tensor and therefore transform appropriately!

H o w e v e r, the specific intensity, the opacity and the emissivity are entirely different 4-animals…

Better Living Through Geometry



Meet the (6-parameter) Lorentz Group

$$\begin{bmatrix} ct' \\ \boldsymbol{x}' \end{bmatrix} = \mathcal{L} \begin{bmatrix} ct \\ \boldsymbol{x} \end{bmatrix}$$

$$\begin{bmatrix} ct \\ \boldsymbol{x} \end{bmatrix} = \mathcal{L}^{-1} \begin{bmatrix} ct' \\ \boldsymbol{x}' \end{bmatrix}$$

$$\gamma = \frac{c}{\sqrt{c^2 - \|\mathbf{u}\|^2}}$$

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\vartheta}) = \begin{bmatrix} \gamma & -\gamma\mathbf{u}/c \\ -\gamma\mathbf{u}/c & \mathbb{R}(\boldsymbol{\vartheta}) + (\gamma - 1)\mathbf{uu}/u^2 \end{bmatrix}$$

$$\mathcal{L}^{-1}(\mathbf{u}, \boldsymbol{\vartheta}) = \mathcal{L}(-\mathbf{u}, -\boldsymbol{\vartheta})$$

$$\mathcal{I} = \mathcal{L}^{-1} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{L}^{-1}$$

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{L}_1 \circ \mathcal{L}_2 \neq \mathcal{L}_2 \circ \mathcal{L}_1$$

Note: This is the 4x4 ***matrix representation*** of the group elements.

There are other representations available.

Please do not confuse the 4x4 ***matrix*** with a ***4-tensor***.

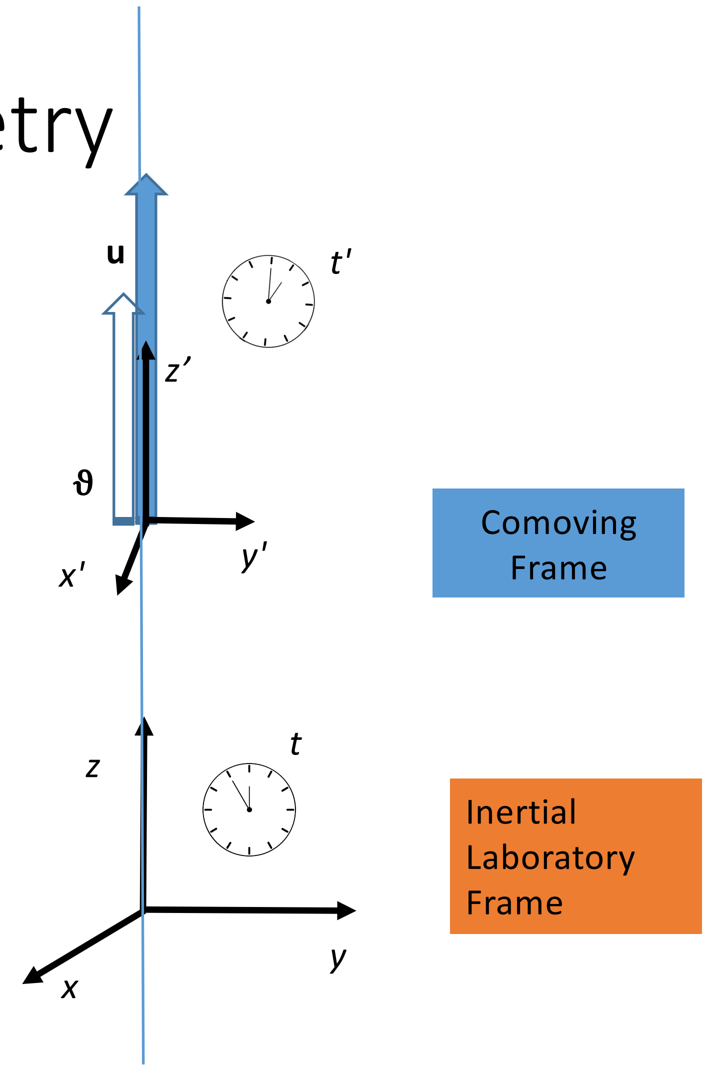
Better Living Through Geometry

These 4-vectors are represented by a 1-column, 4-row matrix!

$$\begin{bmatrix} ct' \\ \boldsymbol{x}' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma u/c \\ 0 & \cos\vartheta & -\sin\vartheta & 0 \\ 0 & \sin\vartheta & \cos\vartheta & 0 \\ -\gamma u/c & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct \\ \boldsymbol{x} \end{bmatrix}$$

$$\begin{bmatrix} ct \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & \gamma u/c \\ 0 & \cos\vartheta & \sin\vartheta & 0 \\ 0 & -\sin\vartheta & \cos\vartheta & 0 \\ \gamma u/c & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} ct' \\ \boldsymbol{x}' \end{bmatrix}$$

Note: Generally speaking-- $\text{Boost}_1 \circ \text{Boost}_2 = \text{Boost}_3 + \text{Rotation}$ unless the Boosts are parallel!



Comoving vs Inertial Frames

$$\begin{bmatrix} \nu' \\ \nu'\mathbf{n}' \end{bmatrix} = \mathcal{L} \begin{bmatrix} \nu \\ \nu\mathbf{n} \end{bmatrix}$$

$$\begin{bmatrix} E' & \mathbf{F}'/c \\ \mathbf{F}'/c & \mathbb{P}' \end{bmatrix} = \mathcal{L} \begin{bmatrix} E & \mathbf{F}/c \\ \mathbf{F}/c & \mathbb{P} \end{bmatrix} \mathcal{L}^{T}$$

$$(\mathcal{L}_1 \circ \mathcal{L}_2)^T = \mathcal{L}_2{}^T \circ \mathcal{L}_1{}^T$$

Note: 4-tensors and 4x4 matrices, 4-vectors and 4x1 or 1x4 matrices, all look deceptively similar to one another, ***but*** it is important to recognize their differences!

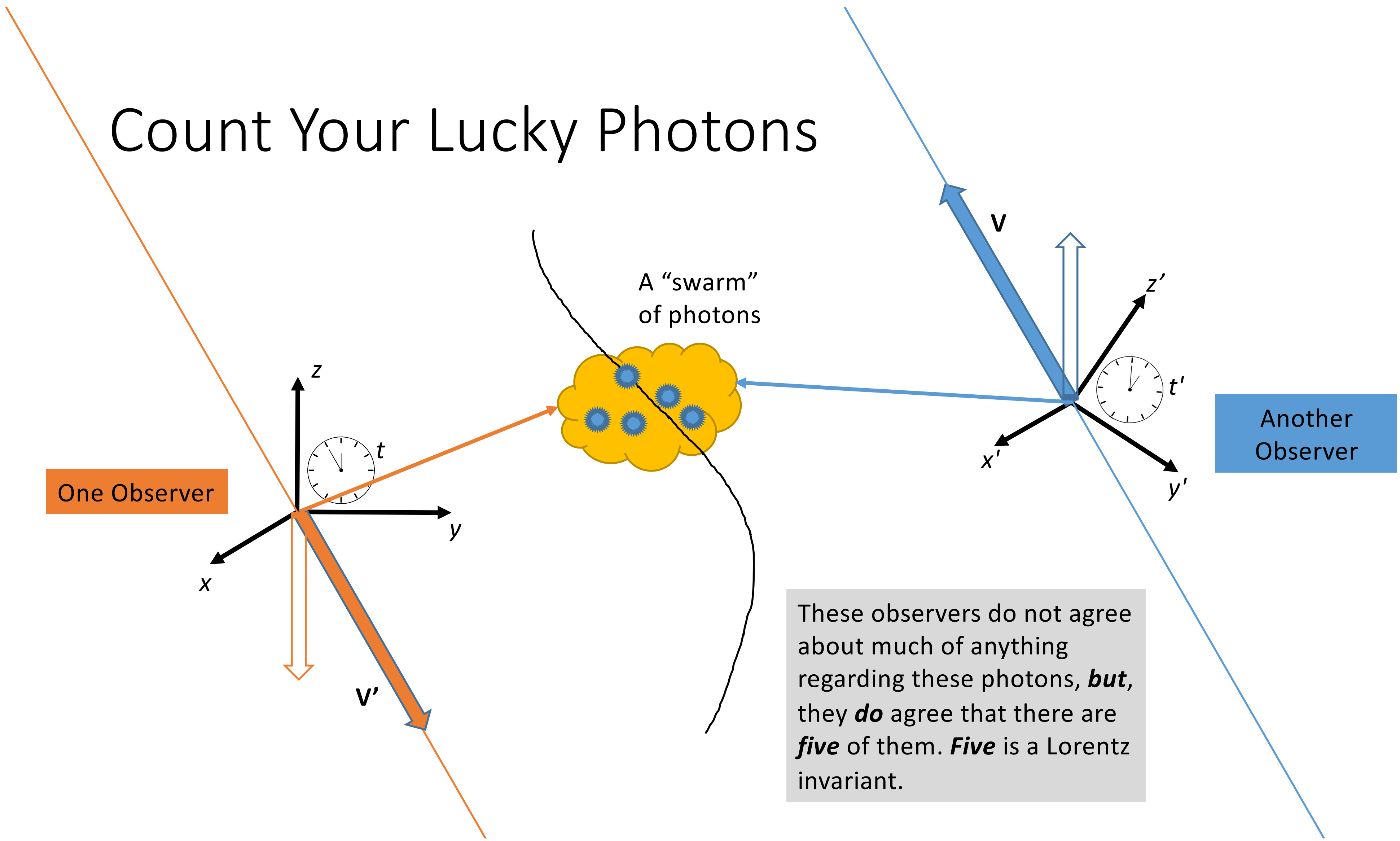
The transpose "T" of a 4x4 matrix exchanges rows with columns.

The transpose "T" of a 1-column, 4-row matrix is a 4-column, 1-row matrix.

$$-L^{\alpha} = [-\boldsymbol{J} \cdot \boldsymbol{E}/c, -\rho_e \boldsymbol{E} - \boldsymbol{J} \times \boldsymbol{B}]^T$$

$$-G^{\alpha} = \frac{1}{c} \int_0^{\infty} d\nu \int d\mathbf{n}\, n^{\alpha} [\eta_\nu - \chi_\nu I_\nu]$$

Count Your Lucky Photons



These observers do not agree about much of anything regarding these photons, ***but***, they ***do*** agree that there are ***five*** of them. ***Five*** is a Lorentz invariant.

Summary

Lorentz Transformation of the Radiation Field

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}$$

$$\nu\chi_\nu = \nu'\chi'_{\nu'}$$

$$\frac{\eta_\nu}{\nu^2} = \frac{\eta'_{\nu'}}{\nu'^2}$$

$$E' = \gamma^2\left[E - \frac{2}{c^2}\boldsymbol{u}\cdot\boldsymbol{F} + \frac{1}{c^2}\boldsymbol{u}\boldsymbol{u}:\mathbb{P}\right]$$

$$\boldsymbol{F}' = \boldsymbol{F} - \boldsymbol{u}[E + \mathbb{P}] + \cdots$$

$$\mathbb{P}' = \mathbb{P} - \frac{1}{c^2}[\boldsymbol{uF} + \boldsymbol{Fu}] + \cdots$$

$$\nu' = \gamma\nu\left(1 - \frac{1}{c}\mathbf{n}\cdot\mathbf{u}\right)$$

$$\mathbf{n}' = \frac{c\mathbf{n} - \gamma\mathbf{u}}{c - \mathbf{n}\cdot\mathbf{u}}\left[\frac{1}{\gamma} - \frac{1}{\gamma+1}\frac{\mathbf{n}\cdot\mathbf{u}}{c}\right]$$

These ***exact*** equations expresses the Doppler shift and aberration of the photons.

The comoving frame (*primed quantities*) has a velocity **u** measured in the inertial laboratory frame (*unprimed quantities*).

$$\gamma = \frac{c}{\sqrt{c^2 - \|\mathbf{u}\|^2}} \approx 1$$

This part is tricky. Remember each frame has its own independent variables: **x**, ct, ν**n**, ν that transform like components of a 4-vector.

Summary

Lorentz Transformation of the Electromagnetic Fields

$$\boldsymbol{E}' = \gamma[\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}] - (\gamma - 1)(\boldsymbol{E} \cdot \boldsymbol{u})\boldsymbol{u} \,/\, \|\boldsymbol{u}\|^2$$

$$\boldsymbol{B}' = \gamma[\boldsymbol{B} - \boldsymbol{u} \times \boldsymbol{E}/c^2] - (\gamma - 1)(\boldsymbol{B} \cdot \boldsymbol{u})\boldsymbol{u} \,/\, \|\boldsymbol{u}\|^2$$

$$\boldsymbol{J}' = \boldsymbol{J} - \gamma \boldsymbol{u}\rho_e + (\gamma - 1)(\boldsymbol{J} \cdot \boldsymbol{u})\boldsymbol{u} \,/\, \|\boldsymbol{u}\|^2$$

$$\rho_e{}' = \gamma[\rho_e - \boldsymbol{u} \cdot \boldsymbol{J}/c^2]$$

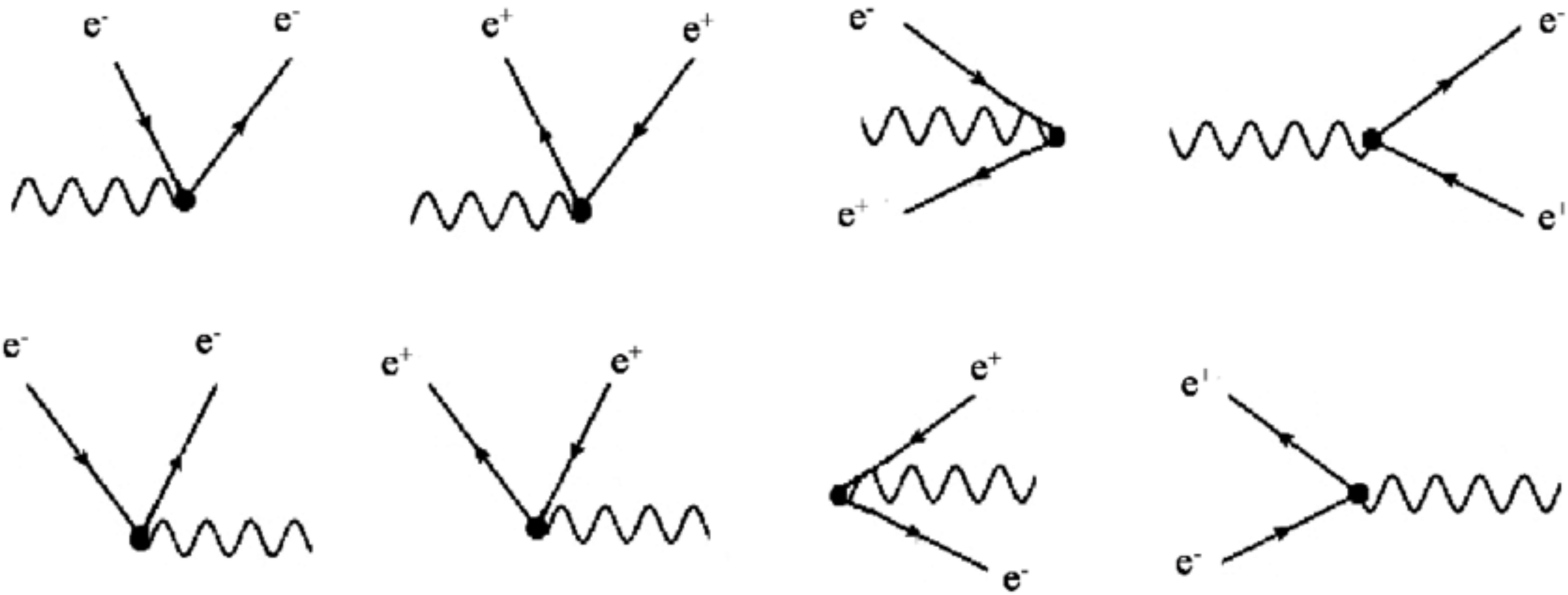
These ***exact*** equations express the transformations of the electromagnetic fields

The comoving frame (*primed quantities*) has a velocity $\mathbf{u}$ measured in the inertial laboratory frame (*unprimed quantities*).

"You had better know how to transform coordinates, physical quantities, fields, differential and integral operators (and anything else you can think of) between ***any*** *two frames of reference, under* ***all*** *conditions."*

$$\gamma = \frac{c}{\sqrt{c^2 - \|\mathbf{u}\|^2}} \approx 1$$

The Interaction Between Radiation & Matter



The Interaction Between Radiation & Matter

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu = \eta_\nu - \chi_\nu I_\nu$$

Let's assume we are in the comoving frame at rest in the laboratory frame.

χ_ν:

- **Destruction** via absorption — $\pi_\nu(\mathbf{n})$
- **Redirection** via scattering
- Frequency **Shift** via scattering — $\sigma(\nu_1, \mathbf{n}_1 \rightarrow \nu_2, \mathbf{n}_2)$

η_ν:

- **Redirection** via scattering
- Frequency **Shift** via scattering — $\sigma(\nu_1, \mathbf{n}_1 \rightarrow \nu_2, \mathbf{n}_2)$
- **Creation** via emission — $S_\nu(\mathbf{n})$
  - Spontaneous
  - Stimulated by I_ν

This is a purely quantum mechanical effect with no real classical counterpart.

The Interaction Between Radiation & Matter

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = \eta_\nu - \chi_\nu I_\nu$$

The true absorption.

$$\left[S_\nu\left(1+\frac{c^2}{2h\nu^3}I_\nu\right) - \pi_\nu I_\nu\right] + \int_0^\infty \mathrm{d}\nu_1 \oint \mathrm{d}\mathbf{n}_1 \frac{\nu}{\nu_1}\sigma(\nu_1,\mathbf{n}_1 \to \nu,\mathbf{n})I_{\nu_1}(\mathbf{n}_1)\left(1+\frac{c^2}{2h\nu^3}I_\nu(\mathbf{n})\right)$$

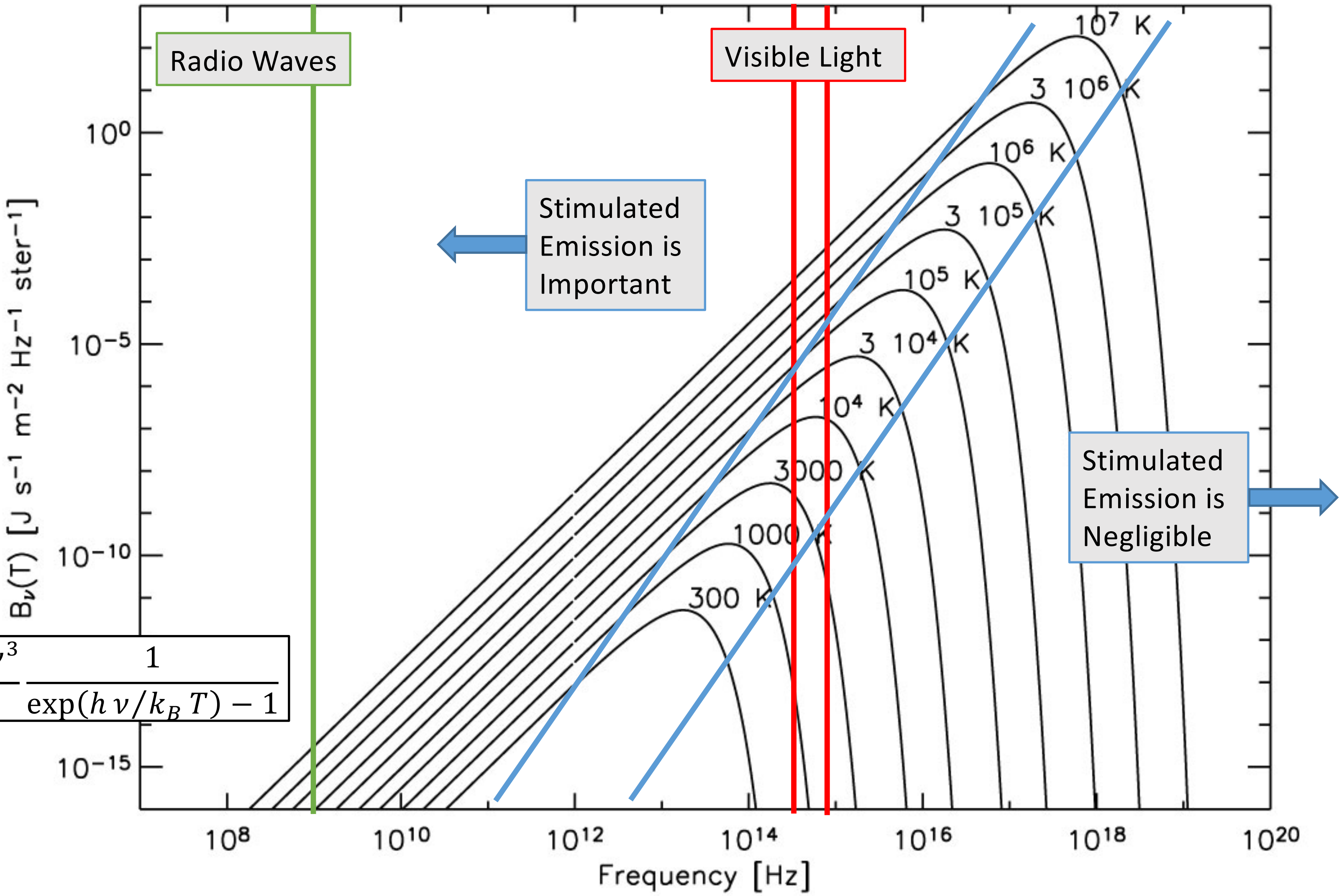
$$-\int_0^\infty \mathrm{d}\nu_1 \oint \mathrm{d}\mathbf{n}_1\,\sigma(\nu,\mathbf{n}\to\nu_1,\mathbf{n}_1)\,I_\nu(\mathbf{n})\left(1+\frac{c^2}{2h{\nu_1}^3}I_{\nu_1}(\mathbf{n}_1)\right)$$

In LTE we must have:

$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T)-1}$$

$$S_\nu = \kappa_\nu B_\nu$$

The absorption ***corrected*** for stimulated emission.



$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\,\nu/k_B\,T)-1}$$

In the Moment(s)

$$E_\nu = \frac{1}{c}\oint \mathrm{d}\mathbf{n}\, I_\nu$$

$$\boldsymbol{F}_\nu = \oint \mathrm{d}\mathbf{n}\,\mathbf{n}\, I_\nu$$

$$\mathbb{P}_\nu = \frac{1}{c}\oint \mathrm{d}\mathbf{n}\,\mathbf{n}\mathbf{n}\, I_\nu$$

$$\mathfrak{Q}_\nu = \oint \mathrm{d}\mathbf{n}\,\mathbf{n}\mathbf{n}\mathbf{n}\, I_\nu$$

Axisymmetry

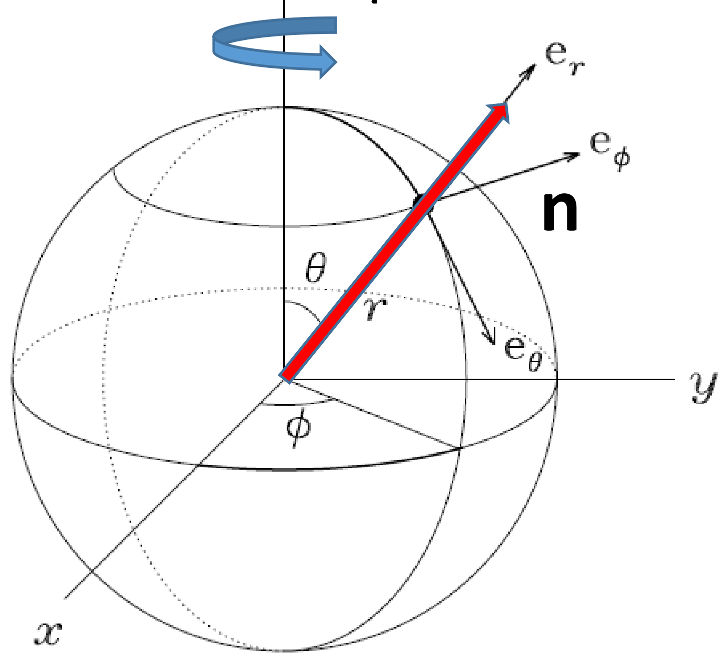
E_ν

F_ν

$P_\nu \quad E_\nu$

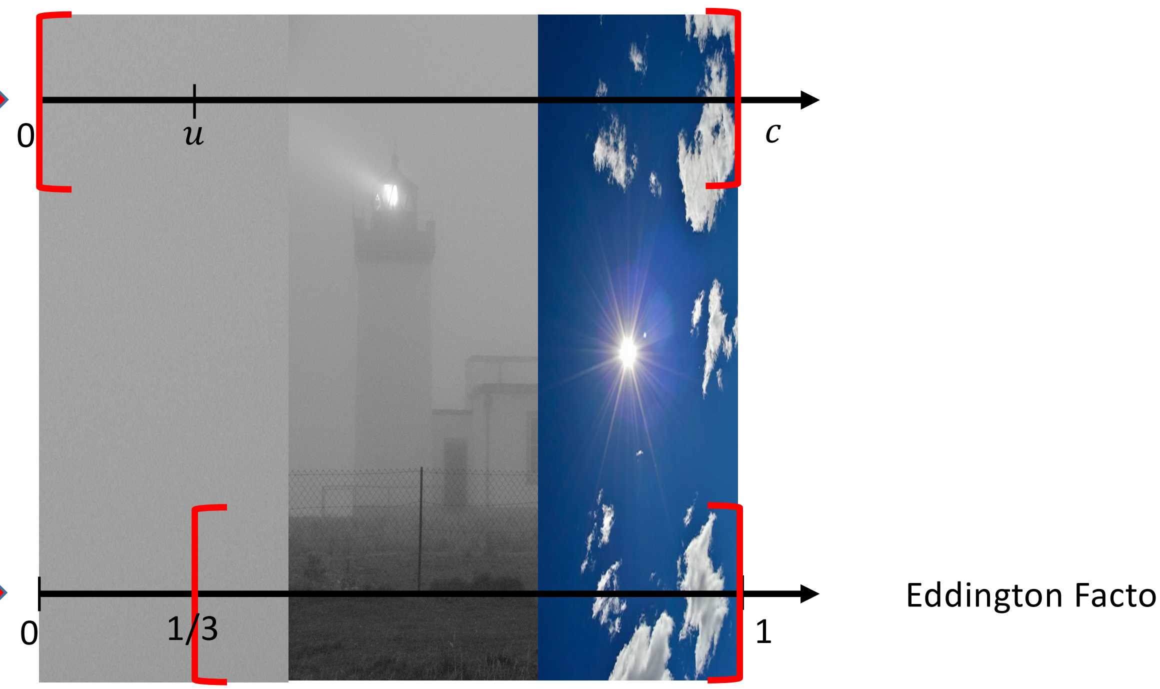
$Q_\nu \quad F_\nu$

$\mu = \cos\theta$



$$\frac{\boldsymbol{F}_\nu = \oint \mathrm{d}\mathbf{n}\,\mathbf{n}\,I_\nu}{E_\nu = \frac{1}{c}\oint \mathrm{d}\mathbf{n}\,I_\nu}$$

$$\frac{\mathbb{P}_\nu = \frac{1}{c}\oint \mathrm{d}\mathbf{n}\,\mathbf{n}\mathbf{n}\,I_\nu}{E_\nu = \frac{1}{c}\oint \mathrm{d}\mathbf{n}\,I_\nu}$$



Diffusion Limit

$$I_\nu \approx \frac{c}{4\pi} E_\nu + \frac{3}{4\pi}\mathbf{n}\cdot\boldsymbol{F}_\nu$$

Ouch!

Free Streaming

$$I_\nu \approx c E_\nu \delta(\mu - 1)$$

Example 1

Gray/LTE Approximation in the *Comoving* Frame

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}$$

$$\nu\chi_\nu = \nu'\kappa$$

$$\frac{\eta_\nu}{\nu^2} = \frac{\kappa B_{\nu'}}{\nu'^2}$$

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T) - 1}$$

A ***constant***, frequency-independent opacity κ, and the ***isotropic*** Planck Function (but frequency dependent) constitutes the "gray atmosphere" LTE ***approximation*** in the comoving frame.

Notice that in the laboratory frame the emissivity and the opacity are ***not*** isotropic because of the Doppler-shifted frequency!

Example 1

Gray/LTE Approximation in the *Laboratory* Frame

$$\eta_\nu = \kappa \left[\left(1 + \frac{3}{c}\,\mathbf{n}\cdot\mathbf{u}\right) B_\nu - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{u}\,\frac{\partial}{\partial \nu}\,\nu B_\nu \right] + \cdots$$

$$\chi_\nu = \kappa \left(1 - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{u}\right) + \cdots$$

This part is subtle. We want the Planck Function evaluated at the laboratory frequency ***not*** the comoving frame frequency.

We can now carry out the two integrals we need to describe the exchange of ***energy*** and momentum between the material and the radiation field in the ***laboratory frame***.

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = \int_0^\infty d\nu \int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

Example 1

Gray/LTE Approximation in the *Laboratory* Frame

$$\eta_\nu = \kappa \left[\left(1 + \frac{3}{c}\mathbf{n}\cdot\mathbf{u}\right) B_\nu - \frac{1}{c}\mathbf{n}\cdot\mathbf{u}\frac{\partial}{\partial\nu}\nu B_\nu\right] + \cdots$$

$$\chi_\nu = \kappa \left(1 - \frac{1}{c}\mathbf{n}\cdot\mathbf{u}\right) + \cdots$$

$$\sigma_R = \frac{2\pi^5 k_B^4}{15 h^3 c^2}$$

Higher-order terms in the ratio of u/c live here.

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = \kappa\left[4\sigma_R T^4 - cE\right] + \kappa\frac{1}{c}\boldsymbol{u}\cdot\boldsymbol{F} + \cdots$$

Example 1

Gray/LTE Approximation in the *Laboratory* Frame

$$\eta_\nu = \kappa \left[\left(1 + \frac{3}{c}\,\mathbf{n}\cdot\mathbf{u}\right) B_\nu - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{u}\,\frac{\partial}{\partial\nu}\,\nu B_\nu \right] + \cdots$$

$$\chi_\nu = \kappa \left(1 - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{u}\right) + \cdots$$

We can now carry out the two integrals we need to describe the exchange of energy and ***momentum*** between the material and the radiation field in the ***laboratory frame***.

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu \int d\mathbf{n}\,\mathbf{n}\left[\eta_\nu - \chi_\nu I_\nu\right]$$

Example 1

Gray/LTE Approximation in the *Laboratory* Frame

$$\eta_\nu = \kappa \left[\left(1 + \frac{3}{c}\mathbf{n}\cdot\mathbf{u}\right) B_\nu - \frac{1}{c}\mathbf{n}\cdot\mathbf{u}\frac{\partial}{\partial\nu}\nu B_\nu \right] + \cdots$$

$$\chi_\nu = \kappa \left(1 - \frac{1}{c}\mathbf{n}\cdot\mathbf{u}\right) + \cdots$$

Higher-order terms in the ratio of u/c live here.

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = -\frac{\kappa}{c}\left[\boldsymbol{F} - \boldsymbol{u}\left\{\frac{4\sigma_R}{c}T^4 + \mathbb{P}\right\}\right] + \cdots$$

Example 1

Gray/LTE Approximation in the *Laboratory* Frame

$$\eta_\nu = \kappa\,[\,(1 + \frac{3}{c}\,\mathbf{n}\cdot\mathbf{u}\,)\,B_\nu - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{u}\,\frac{\partial}{\partial\nu}\,\nu B_\nu] + \cdots$$

$$\chi_\nu = \kappa\,(1 - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{u}\,) + \cdots$$

How else could we have done this?
Hint: do you know any other 4-vectors?

Higher-order terms in the ratio of u/c live here.

$$\frac{1}{c^2}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = -\frac{\kappa}{c}[\,\boldsymbol{F} - \boldsymbol{u}\{\frac{4\sigma_R}{c}T^4 + \mathbb{P}\}] + \cdots$$

Example 2

Dynamic vs Static Diffusion

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{F} = \kappa\,[4\sigma_R T^4 - cE] + \kappa \frac{1}{c}\,\boldsymbol{u} \cdot \boldsymbol{F} + \cdots$$

$$\frac{1}{c^2} \cdot \frac{\partial \boldsymbol{F}}{\partial t} + \nabla \cdot \mathbb{P} = -\frac{\kappa}{c}[\,\boldsymbol{F} - \boldsymbol{u}\{\frac{4\sigma_R}{c} T^4 + \mathbb{P}\}] + \cdots$$

Assume: sufficiently small mean free path so the radiation pressure tensor is isotropic to leading order.

$$\begin{cases} P \approx \dfrac{1}{3} E \\ \boldsymbol{F} \approx \boldsymbol{u}\left(\dfrac{4\sigma_R}{c} T^4 + \dfrac{1}{3} E\right) - \dfrac{c}{3\kappa} \nabla E \end{cases}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\boldsymbol{u}\left(\frac{4\sigma_R}{c} T^4 + \frac{1}{3} E\right) - \frac{c}{3\kappa} \nabla E\,\right] \approx \kappa\,[4\sigma_R T^4 - cE] - \frac{1}{3}\boldsymbol{u} \cdot \nabla\, E$$

Example 2

Dynamic vs Static Diffusion

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \left[\boldsymbol{u}\left(\frac{4\sigma_R}{c}T^4 + \frac{1}{3}E\right) - \frac{c}{3\kappa}\boldsymbol{\nabla}E\right] = \kappa\left[4\sigma_R T^4 - cE\right]$$

Assume: sufficiently small mean free path so this term is the dominant balance in the radiation energy equation.

1 $\quad u\lambda/c\ell \quad \lambda^2/\ell^2$

$$E \approx \frac{4\sigma_R T^4}{c} - \frac{4\sigma_R}{c^2\kappa}\left[\frac{\partial}{\partial t} - \frac{1}{3}\boldsymbol{u}\cdot\nabla + \nabla\cdot\left(\frac{4}{3}\boldsymbol{u} - \frac{c}{3\kappa}\boldsymbol{\nabla}\right)\right]T^4$$

$u/c \quad \lambda/\ell$

$$P \approx \frac{1}{3}E$$

$$\boldsymbol{F} \approx \boldsymbol{u}\left(\frac{4\sigma_R}{c}T^4 + \frac{1}{3}E\right) - \frac{c}{3\kappa}\boldsymbol{\nabla}E$$

Example 2

Dynamic vs Static Diffusion

$$\chi_r = \frac{16\sigma_R T^3}{3\kappa}$$

$$\boldsymbol{F} \approx \frac{4\sigma_R}{c}\left[\left(\frac{4}{3}\boldsymbol{u}T^4 - \frac{4cT^3}{3\kappa}\boldsymbol{\nabla}\, T\right)\right]$$

u $c\lambda/\ell$

u/c λ/ℓ

$$\frac{\partial}{\partial t}e + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{1}{\rho}\boldsymbol{\nabla}\cdot[(\chi + \chi_r)\boldsymbol{\nabla}T]$$

Conservation de l’énergie interne (1.56)

Example 2

Dynamic vs Static Diffusion

$$\boldsymbol{F} \approx \frac{4\sigma_R}{c}\left[\left(\frac{4}{3}\boldsymbol{u}T^4 - \frac{4cT^3}{3\kappa}\boldsymbol{\nabla}\, T\right)\right]$$

u $\quad$ $c\lambda/\ell$

u/c $\quad$ λ/ℓ

$$\frac{\partial}{\partial t}e + \boldsymbol{u}\cdot\boldsymbol{\nabla}e + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{1}{\rho}\boldsymbol{\nabla}\cdot\left[\chi\boldsymbol{\nabla}T - \frac{16\sigma_R}{3c}\boldsymbol{u}T^4\right]$$

La nouvelle conservation de l'énergie interne (1.56)'

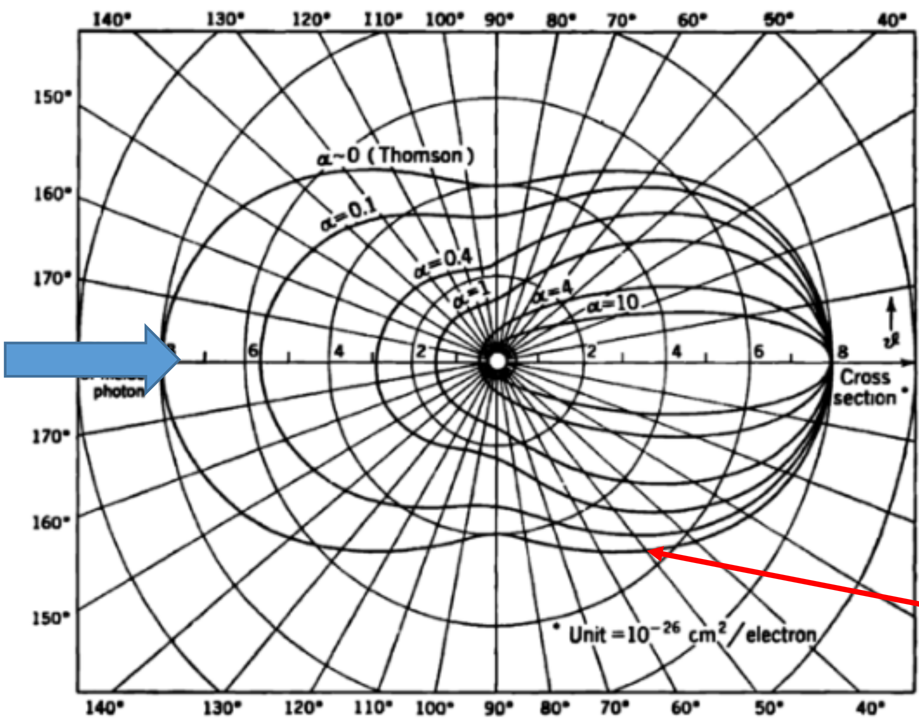
Example 3

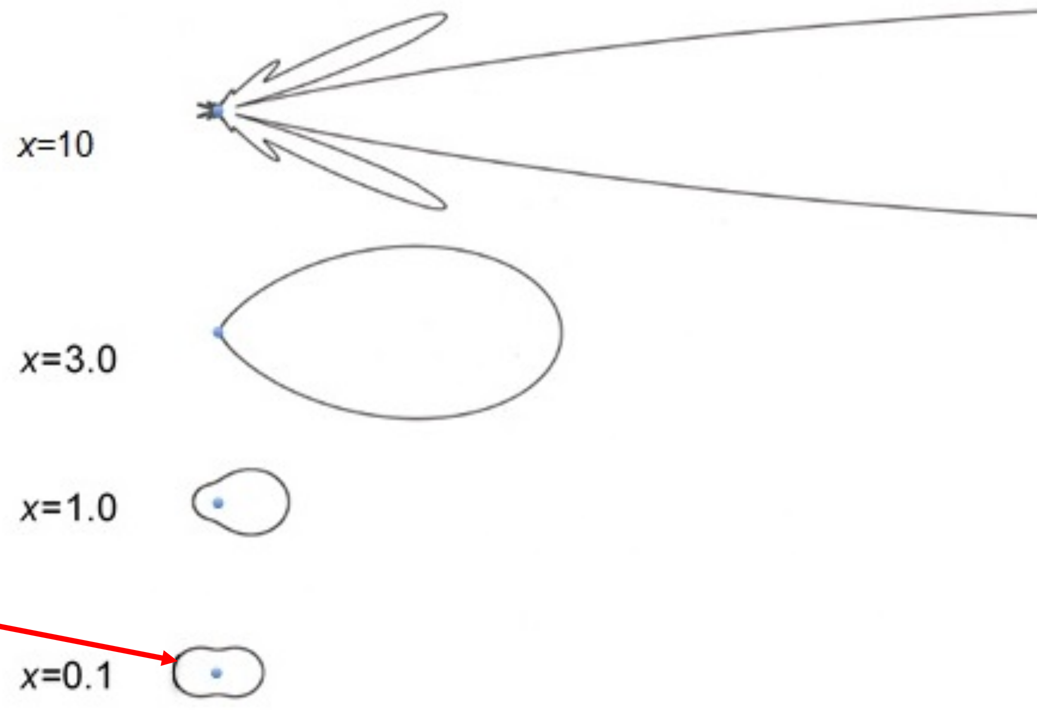
Scattering

Monopole Scattering
(Thomson/Compton)

$$\sigma(\nu_1, \mathbf{n}_1 \rightarrow \nu_2, \mathbf{n}_2)$$

Dipole Scattering
(Rayleigh/Mie)





Example 3

Isotropic Scattering (Thomson)

$$\frac{1}{c}\cdot\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = \gamma\rho[J_\nu - I_\nu]$$

$$J_\nu = \frac{1}{4\pi}\oint d\mathbf{n}\, I_\nu$$

$$\gamma = \frac{8\pi}{3m_H}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \approx 0.04\ \mathrm{m^2/kg}$$

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = 0 \quad\Rightarrow\quad F = \frac{\mathcal{L}_{\mathrm{rad}}}{4\pi r^2}$$

$$\frac{1}{c}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + c\,\nabla\cdot\mathbb{P} = -\gamma\rho\boldsymbol{F}$$

$$\nabla\cdot\mathbb{P} = \frac{\mathrm{d}P_{rr}}{\mathrm{d}r} + \frac{2P_{rr} - P_{\theta\theta} - P_{\phi\phi}}{r}$$

$$\mathrm{Tr}\,\mathbb{P} = E \qquad E = P_{rr} + P_{\theta\theta} + P_{\phi\phi}$$

B.2.24 on page 200

We can assume $P_{\theta\theta} = P_{\phi\phi}$ but we are still one equation short.

Example 3

Isotropic Scattering (Thomson)

$$\mathbf{n}\cdot\nabla I_\nu = \gamma\rho[J_\nu - I_\nu]$$

$$J_\nu = \frac{1}{4\pi}\oint d\mathbf{n}\, I_\nu$$

$$\gamma = \frac{8\pi}{3m_H}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 \approx 0.04\ \mathrm{m^2/kg}$$

$$\nabla\cdot\boldsymbol{F} = 0 \;\Rightarrow\; F = F_{\mathrm{rad}}$$

$$c\,\nabla\cdot\mathbb{P} = -\gamma\rho\boldsymbol{F}$$

$$\nabla\cdot\mathbb{P} = \frac{\mathrm{d}P_{zz}}{\mathrm{d}z}$$

$$\mathrm{Tr}\,\mathbb{P} = E \qquad E = P_{zz} + P_{xx} + P_{yy}$$

We can assume $P_{xx} = P_{yy}$ but we are still one equation short.

Example 3

Isotropic Scattering (Planar Geometry)

$$\mathbf{n} \cdot \boldsymbol{\nabla} I_\nu = \gamma\rho[J_\nu - I_\nu]$$

$$J_\nu = \frac{1}{4\pi}\oint d\mathbf{n}\, I_\nu$$

$$F = F_{\text{rad}}$$

$$f_{\text{Edd}} = \frac{cP_{zz} = F_{\text{rad}}\ [\tau + q(\infty)]}{cE = 3F_{\text{rad}}\ [\tau + q(\tau)]}$$

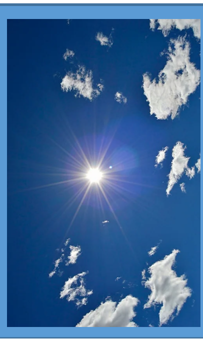
$q(0) = 0.5773 \ldots$

$q(1) = 0.6985 \ldots$

$q(\infty) = 0.7104 \ldots$

0.333... 0.410...

0.999...



Spherical Geometry

Example 4

Rayleigh & Thomson Scattering

$$\frac{1}{c}\cdot\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = \varpi_\nu \rho \frac{3}{4}\left[J_\nu + \mathbf{nn}\colon \mathbb{K}_\nu - \frac{4}{3} I_\nu\right]$$

$$\mathbb{K}_\nu = \frac{1}{4\pi}\oint d\mathbf{n}\,\mathbf{nn}\, I_\nu$$

$$J_\nu = \frac{1}{4\pi}\oint d\mathbf{n}\, I_\nu$$

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = ?$$

$$\frac{1}{c}\cdot\frac{\partial \boldsymbol{F}}{\partial t} + c\,\nabla\cdot\mathbb{P} = ?$$

Example 5

Isotropic Scattering in the *Comoving* Frame

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}$$

$$\nu\chi_\nu = \nu'\sigma$$

$$\frac{\eta_\nu}{\nu^2} = \frac{\sigma J'_{\nu'}}{\nu'^2}$$

$$\sigma(\nu_1, \mathbf{n}_1 \to \nu_2, \mathbf{n}_2)$$

$$J_\nu = \frac{1}{4\pi} \oint d\mathbf{n}\, I_\nu$$

A ***constant***, frequency-independent opacity, and the ***mean intensity*** (but frequency dependent) as source function constitutes the isotropic ***approximation*** of Thomson scattering.

Notice that in the laboratory frame the emissivity and the opacity are ***not*** isotropic because of the Doppler shifted frequency!

Example 5

Isotropic Scattering in the *Comoving* Frame

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}$$

$$\nu\chi_\nu = \nu'\sigma$$

$$\frac{\eta_\nu}{\nu^2} = \frac{\sigma J'_{\nu'}}{\nu'^2}$$

?

$$\sigma(\nu_1, \mathbf{n}_1 \rightarrow \nu_2, \mathbf{n}_2)$$

$$J_\nu = \frac{1}{4\pi}\oint d\mathbf{n}\, I_\nu$$

Notice that in the laboratory frame the emissivity and the opacity are ***not*** isotropic because of the Doppler shifted frequency!

Your HOMEWORK Assignment!

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = \int_0^\infty d\nu \int d\mathbf{n}\,[\eta_\nu - \chi_\nu I_\nu]$$

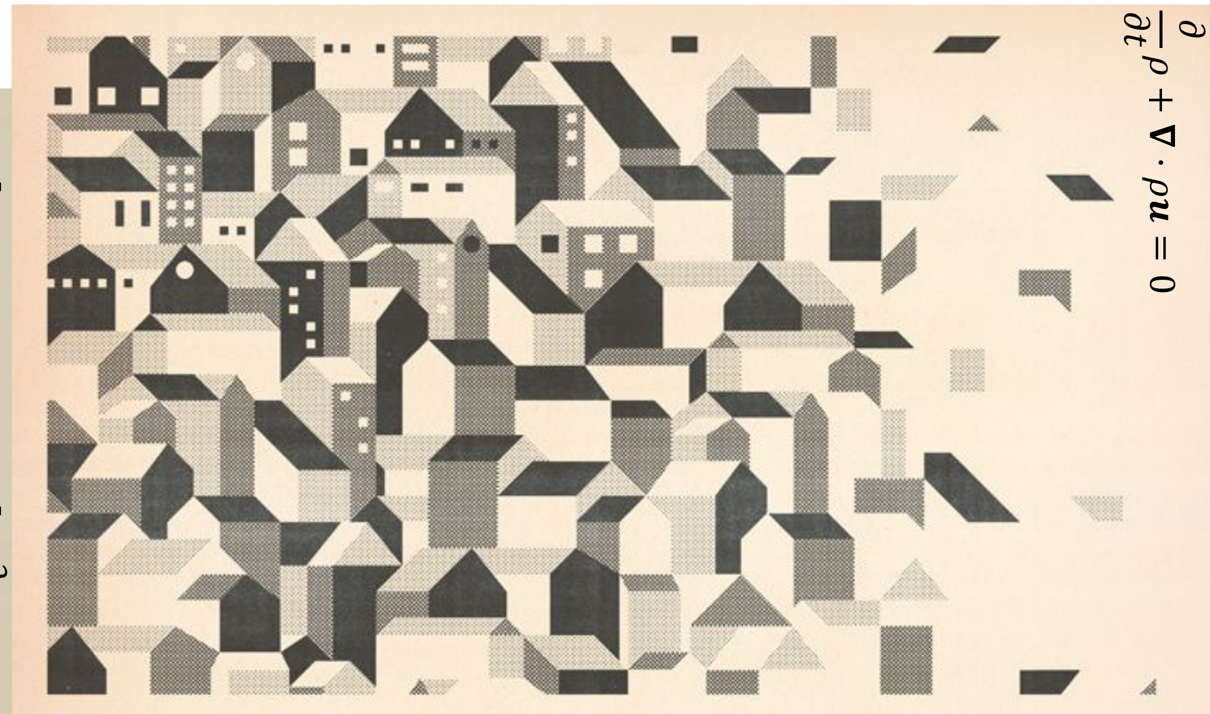
$$\frac{1}{c^2}\frac{\partial \boldsymbol{F}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbb{P} = \frac{1}{c}\int_0^\infty d\nu \int d\mathbf{n}\,\mathbf{n}[\eta_\nu - \chi_\nu I_\nu]$$

Astrophysical Radiation Magnetohydrodynamics

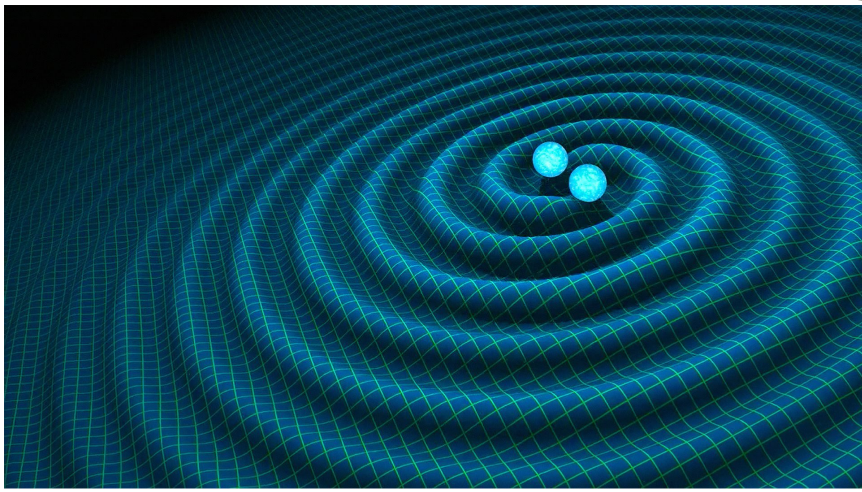
Being an Opera in Four Acts

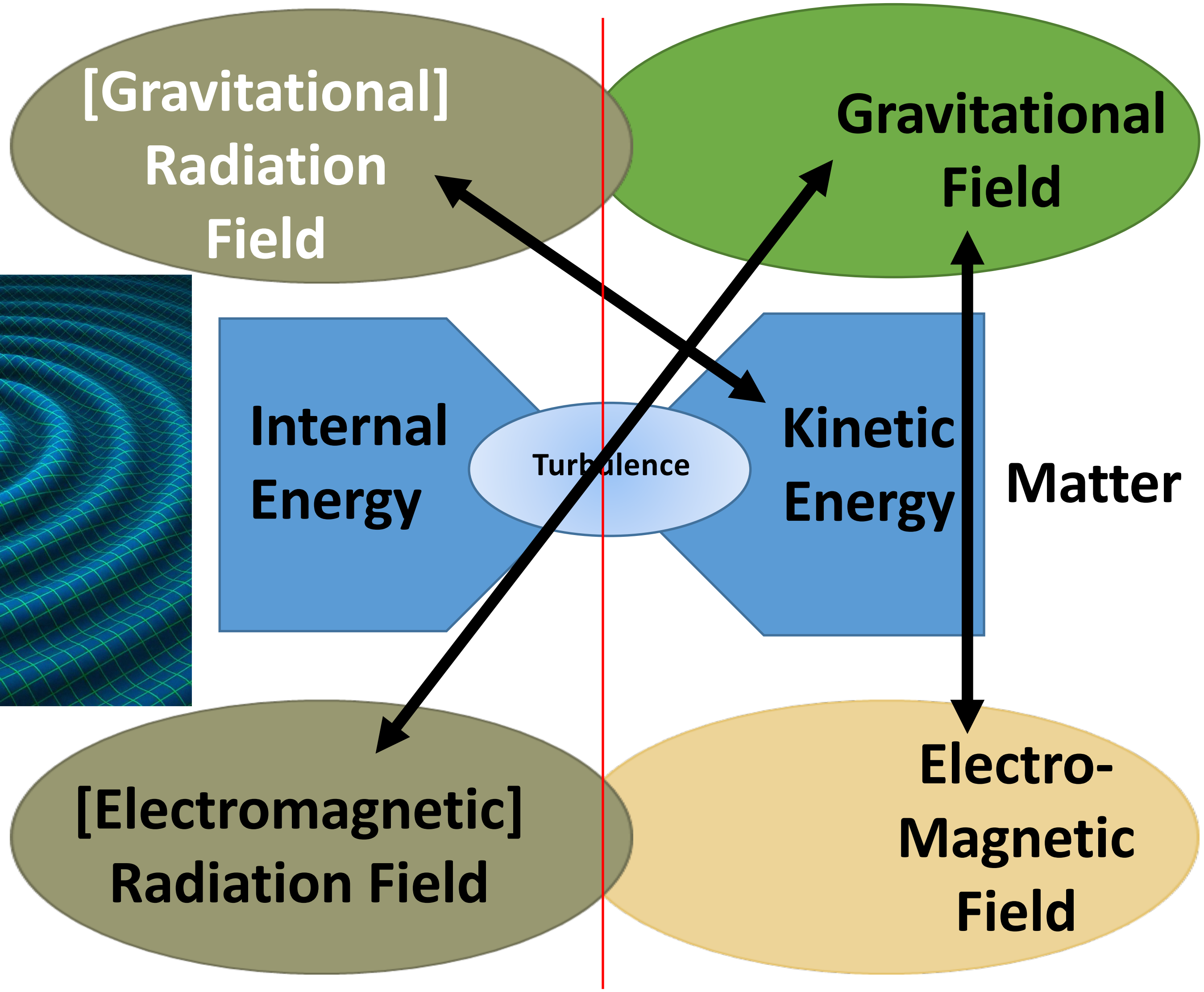
CURTAIN
CALL

Construction vs *Deconstruction*



A Final Thought…!





Next Lecture…


To be
continued…