PHY 3700 meets PHY 6756

Lecture 1: Radiation Magnetohydrodynamics

Lecture 2: Spherically-Symmetric Applications

NOTES & COMMENTS

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It always pays to read the fine print.

Preface

Here is the so-called "fine print" that supports (to some degree) the material presented in the slides. It provides an opportunity for you to gain some deeper insights about what has been presented, understand some of the essential features of the derivations that were not presented, and head off in different directions to explore on your own.

I have tried throughout to refer to equations or material in your possession from the lecture notes which accompany PHY3700 and PHY6756. For example, the expression "(3700:1.25)" refers to equation (1.25) in the lecture notes which accompany PHY3700. I have tried, as much as humanly possible (for me, at least) to leave the notation as you discovered it in PHY3700 and PHY6756. However, notation is more than just notation. Poor notation obfuscates and excellent notation promotes quick comprehension. In several cases I could not help myself and I apologize for my stubbornness in advance. Where it is not entirely obvious I have tried to indicate how my notation differs from what you are used to.

On this point it is worth stating here that we have an insufficient number of symbols for the physical quantities we wish to talk about. This is especially true when radiative transfer and MHD are mixed---J's and H's and B's and E's end up being used for different things in each discipline. One faces the daunting problem of creating an even greater proliferation of symbols and fonts and superscripts or just going with the flow and hoping that context serves to disambiguate confusion.

SI versus cgs/esu for the description of the electromagnetic field is a sore point with almost every practicing physicist on the planet. Like cats and dogs, we shall be living with this conflict through out the rest of our lives. My advice is to be aware of how to convert between the two systems (useful formulae are provided below) as needed. Do your theory in cgs/esu and build your experiments in SI. In the former know that the ubiquitous presence of 3.14159… has to do with solid angles hidden all over the place, and in the latter rejoice that free space (vacuum) has a nonzero permittivity and permeability, however that might be possible even without virtual pair production.

The items in the Bibliography are cited by bolding the first three letters of the lead author's last name and appending the publication date, as in **Arn1996**, for example. I do not cite every one of the references presented to you in the Bibliography. The reason is simple. I have found over the years that there is a set of references that I continually find myself returning to over and over again. To me, each is particularly useful for something that I can't seem to find nowhere else. Often it is merely a matter of taste. I recommend that you assemble a similar set as you progress with your careers. For this reason the Bibliography is in no way complete, nor is it exhaustive. The material is also very uneven---some is quite elementary, and some is incredibly complex.

In some of the comments, I make some suggestions for calculations you might want to try out, or issues you could profitably explore based on the material presented in these two lectures. As my Dad once said, you now know enough to really get into trouble. But, with enough effort, you can always get yourself out of trouble, with the knowledge from a great experience to add to your arsenal of capabilities.

Finally, I apologize for errors and typos that have drifted into this material and my understanding of the physics. Trust but verify is a good motto to follow. Please do let me know if you find anything egregiously wrong or point out misprints.

Cheers,

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The Radiation Field

The full radiative transfer equation (which ***is*** Lorentz invariant) that we require, has an additional term which is absent from equation (3700:2.5.2), and is proportional to the time derivative of the specific intensity. This term is necessary to track radiation fronts, or bursts of photons that free-stream (in vacuo) away from some source, for example. Often, but not always, the time scale for these phenomena is so short compared to typical fluid time scales that his term can be safely neglected even in time-dependent situations. It is for these practical reasons that this term is absent in (3700:2.5.2) and the subject matter that follows therein.

However, from a purely pedagogical and bookkeeping perspective of accounting for the flow of energy and momentum within the radiation field and between the radiation field and the material, it is essential that we retain this term for the present. It is also the necessary addition to the transfer equation to ensure that photons do travel at the speed of light. The second amendment we make to (3700:2.5.2) is to write *d/ds* out explicitly as **n·∇**. In (3700:2.5.2) it is implicitly understood that the coordinate, *s*, is in the direction of propagation of the stream of photons. Accordingly, (3700:2.5.2) must be solved along all possible directions to build up the full specific intensity from the sources and sinks of photons on the right side of this equation.

As noted in PHY 3700, physical units are particularly helpful to eliminate mistakes in copying and writing down equations. We'll denote the physical units of a quantity by placing square brackets "[ ]" around the quantity. For example [see (3700:2.1.2), (3700:2.4.2)]:

$[\, I_\nu \,] = \text{erg cm}^{-2}\,\text{sec}^{-1}\,\text{Hz}^{-1}\,\text{ster}^{-1}$ $\qquad [\, \chi_\nu \,] = \text{cm}^{-1}$

$[\, \eta_\nu \,] = \text{erg cm}^{-3}\,\text{sec}^{-1}\,\text{Hz}^{-1}\,\text{ster}^{-1}$ $\qquad [\, \nu \,] = \text{Hz}$

To complete the picture, it is necessary to supply the opacity, χ_ν, and the emissivity, η_ν. A substantial portion of PHY 3700 is devoted to the microphysics of atoms and molecules necessary to determine these quantities. In RMHD, the essential concept is that these quantities should be determined in the ***co-moving (rest) frame*** of the fluid. In this frame, we expect that the general findings of Chapters 1 and 4 of PHY 3700 ought to apply to aid in the determination of the source function and the opacities.

In any event, the full I_ν, usually contains way more information than we want (or actually require) for our program. We are interested in frequency-integrated quantities like (e.g., [3700:2.1.22]):

$$I(\boldsymbol{x},t;\mathbf{n}) \equiv \int_0^\infty d\nu\, I_\nu(\boldsymbol{x},t;\mathbf{n})$$

$$\eta(\boldsymbol{x},t;\mathbf{n}) \equiv \int_0^\infty d\nu\, \eta_\nu(\boldsymbol{x},t;\mathbf{n})$$

as well as angular moments, like (compare with equations [3700:2.1.23], [3700:2.1.24])):

$$J_\nu(\boldsymbol{x},t) \equiv \frac{1}{4\pi}\int d\mathbf{n}\, I_\nu(\boldsymbol{x},t;\mathbf{n})$$

$$\boldsymbol{H}_\nu(\boldsymbol{x},t) \equiv \frac{1}{4\pi}\int d\mathbf{n}\;\mathbf{n}\, I_\nu(\boldsymbol{x},t;\mathbf{n})$$

$$\mathbb{K}_\nu(\boldsymbol{x},t) \equiv \frac{1}{4\pi}\int d\mathbf{n}\,\mathbf{n}\mathbf{n}\, I_\nu(\boldsymbol{x},t;\mathbf{n})$$

Notice that J_ν, is a *scalar*, $\boldsymbol{H}_\nu$, is a *vector*, and $\mathbb{K}_\nu$ is a *tensor*. If we multiply these equations by 4π steradians (ster), and divide by the speed of light *c*, as necessary, we obtain (cf. [3700:2.1.20]) the radiation energy density, radiation energy flux, and the radiation pressure tensor (see also [3700:2.1.20], [3700:2.1.25]):

$$E = \frac{4\pi}{c}J \qquad \boldsymbol{F} = 4\pi\boldsymbol{H} \qquad \mathbb{P} = \frac{4\pi}{c}\mathbb{K}$$

$[\, E \,] = \text{erg cm}^{-3}$ $\quad [\, \mathbf{F} \,] = \text{erg cm}^{-2}\,\text{sec}^{-1}$ $\quad [\, \mathbb{P} \,] = \text{erg cm}^{-3}$

By integrating the transfer equation over frequency and subsequently taking successive angular moments with respect to powers of **n**, we arrive at a continuum, or "fluid" description of the radiation field, or equivalently, photon gas. In fact, this sequence of equations already has the desired property of being in the "conservation" form our program calls for. The zeroth and first angular moments are the two equations at the very bottom of this slide.

Each successive equation in this sequence of moments equates the time derivative of a given moment to the divergence of the next higher-order moment, as well as the contributions of the emission and absorption at each moment level.

To be actionable (i.e., useful) this system of equations must be truncated at some moment level. Typically, one might try to express the radiation pressure tensor $\mathbb{P}$ in terms of the energy density (the so-called Eddington approximation is $\mathbb{P} = ⅓\, E\, \mathbb{I}$, where $\mathbb{I}$ is the unit tensor). Such a closure is not automatic, and its success generally requires some careful thought and a priori knowledge about the anticipated behavior of the radiation field. The material fluid equations, of course, require the very same closure that is embodied in the so-called equation of state, which relates the gas pressure to the mass and internal energy densities. In some sense, the distinction between these two closure schemes is that usually ions, electrons, atoms and molecules that make up the material collide frequently with one another on macroscopic time scales leading to a statistically steady situation that can be described by a simple equation of state. Photons do not collide with one another (at least, not to any great extent), but rather are forced to move toward a statistical equilibrium through their frequent encounters with the material through emission and absorption. If the photon mean free path λ (essentially the reciprocal of the opacity) is large compared to a typical fluid length scale *l*, then there will generally be an insufficient number of emissions and absorptions to force a (local) statistical equilibrium. Hence, one does not expect any simple (i.e., universal and local) closure relationship between the radiation pressure tensor and the radiation energy density in such cases.

Many of the books on radiation/radiation hydrodynamics are ultimately concerned with the closure issue. Variable Eddington Factors offer one avenue. Especially in situations where the radiation field goes from being optically thick to optically thin, it may be necessary to solve the original transfer equation directly.

The relative size and therefore importance of the terms in the transfer equation, and its moments, can be estimated in terms of two dimensionless parameters, *u/c*, the ratio of the typical material velocity to the speed of light, and *l /λ,* the ratio of typical length scale *l* of the material to the photon mean free path *λ*./**1**/

$l/\lambda \ll u/c \ll 1$: ***Free Streaming***: The photons are virtually decoupled from the material. It is pointless to carry the radiation field along in the problem, and a direct solution of the transfer equation along characteristics is preferred to a continuum fluid description. There is no sensible closure relation for the tensor $\mathbb{P}$. Basically, the entire right side of the equations are negligible.
$u/c \ll l/\lambda \ll 1$: ***Streaming***: The *u*-independent terms on the right side of the equations are now more important than the time derivatives on the left side. The flux divergence terms are dominant. The *u*-dependent terms on the right sides are entirely negligible.
$1 \ll l/\lambda \ll c/u$: ***Static Diffusion***: The time derivatives on the left are entirely negligible. The *u*-independent terms on the right side of both equations are now dominant. Away from boundaries, the Eddington Approximation is a good closure option.
$1 \ll c/u \ll l/\lambda$: ***Dynamic Diffusion***: The *u*-dependent terms on the right sides are now more important than the flux divergence terms on the left. The time derivative on the left of the momentum equation (but not necessarily the energy equation) remains entirely negligible. The Eddington Approximation is an even better closure scheme. The photons are so closely coupled to the material that they essentially behave like an additional (massless) ideal gas with a γ of 4/3.

/**1**/Any of the books on radiation hydrodynamics cited above walk through these scalings in their own particular fashion. I like the discussion in **Mih1984**. I was fortunate to attend Dimitri Mihalas's lectures at the University of Colorado in 1983 as he was preparing the book cited above. I have a copy of his hand-written lecture notes that are in some places, like this topic of relative importance of terms, a lot clearer than the final published monograph. If you'd like a copy of these notes, we can supply it to you. Another very impressive teacher and a brilliant and gentle man.

The Electromagnetic Field

Turning to the electromagnetic field, in SI physical units, the Maxwell Equations are provided in Chapter 2 of PHY 6756 (cf. [6756:2.1-4]):

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{E}} = \frac{1}{\varepsilon_0}\tilde{\rho}_e \qquad \boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} = -\frac{\partial \tilde{\boldsymbol{B}}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{B}} = 0 \qquad \boldsymbol{\nabla} \times \tilde{\boldsymbol{B}} = \mu_0 \tilde{\boldsymbol{J}} + \mu_0\varepsilon_0 \frac{\partial \tilde{\boldsymbol{E}}}{\partial t}$$

where $c^2 = 1/\mu_0\varepsilon_0$. We write them here adorned with tildes (which are absent in PHY 6756) to distinguish them from the cgs/esu quantities, which shall be unadorned. The conversion to cgs/esu units is best accomplished simply by defining new cgs/esu fields, charge densities and electric current densities (distinguished here from their SI counterparts with tildes) using the following prescription:

$$\tilde{\boldsymbol{E}} = \frac{1}{\sqrt{4\pi\varepsilon_0}}\boldsymbol{E} \qquad \tilde{\boldsymbol{B}} = \sqrt{\frac{\mu_0}{4\pi}}\boldsymbol{B}$$

$$\tilde{\rho}_e = \sqrt{4\pi\varepsilon_0}\,\rho_e \qquad \tilde{\boldsymbol{J}} = \sqrt{4\pi\varepsilon_0}\,\boldsymbol{J}$$

Substituting these formulae directly into the SI Maxwell Equations above immediately yields their cgs/esu counterparts:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho_e \qquad c\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad c\boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi\boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t}$$

The physical units of these quantities are as follows:
[**E**] = [**B**] = $gm^{1/2}\,cm^{-1/2}\,sec^{-1}$ = Gauss = Statvolts cm^{-1}
[ρ_e] = $gm^{1/2}\,cm^{-3/2}\,sec^{-1}$ = esu cm^{-3}
[**J**] = $gm^{1/2}\,cm^{-1/2}\,sec^{-2}$ = esu $cm^{-2}\,sec^{-1}$

Starting from Maxwell's Equations, we can construct three conservation laws as follows./**2**/ ***First***, take the divergence of Ampère's Law and use Gauss's Law to arrive at the conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

Second, take the dot product of Ampère's Law with the electric field and subtract the dot product of Faraday's Law with the magnetic field, to provide the conservation of energy (as seen from the perspective of the electromagnetic fields).

The energy density in the electromagnetic fields is proportional to the sum of the squares of the electric and magnetic field vectors./**3**/ The energy flux, ***S*** is commonly referred to as the ***Poynting Vector***. The dot product of the electric current density and the electric field can then be interpreted as the rate at which energy is exchanged between the electromagnetic field and the fluid (which after all supports the electric current density). ***Finally***, to obtain the third conservation law (for the momentum carried by the electromagnetic field), start with the Lorentz force (in cgs/esu units)/**4**/, which we will denote by ***L,*** and use Gauss's Law to eliminate the charge density and Ampère's Law to eliminate the electric current density. Making use of Faraday's Law and Gauss's Law for the magnetic field we arrive, after some vector identities at the desired result.

$$\mathbb{M} \equiv \frac{1}{8\pi}(\|\boldsymbol{E}\|^2 + \|\boldsymbol{B}\|^2)\mathbb{I} - \frac{1}{4\pi}(\boldsymbol{EE} + \boldsymbol{BB})$$

Notice that the momentum density carried by the electromagnetic field is simply the Poynting Flux (divided by c^2), the momentum flux density is the Maxwell Stress Tensor $\mathbb{M}$, and the Lorentz force ***L*** describes the exchange of momentum with the fluid (which, again, carries the electric current density and supports the net build-up of electric charge). To this point, we have made no approximations, thus our equations are Lorentz invariant./**5**/

/**2**/Many authors describe this construction, e.g.., **Ogi2016**, **Rob1967**, **Kul2005**, take your pick. Here, I am following **Par1979**. Again, this can also be accomplished beginning directly with the MHD approximation, resulting in the absence of several terms (usually proportional to $\|\boldsymbol{E}\|^2$ and ρ_e) that appear in our equations, and the explicit appearance of the fluid velocity ***u***.

/**3**/I am using the awkward notation of $\|\boldsymbol{E}\|^2$ for the magnitude of the electric field vector, rather than simply E^2 to distinguish it from the square of the energy density of the radiation field, which, would also be E^2. Ditto for ***J*** versus *J* (MHD electric current density versus mean intensity of the radiation field). The unfortunate lack of distinct symbols for key quantities reflects the fact that radiation hydrodynamics and MHD have grown up without much contact to date. To be consistent, I should put all vector lengths in such a format, but, I will be lazy with things like the fluid velocity ***u***, and just write u^2 say, because there is no danger in confusing it with something else.

To close these equations and render them actionable, it is necessary to relate (in some fashion) the electric charge density ρ_e and the electric current density ***J*** to the electromagnetic fields. Like RMHD, the essential concept is again that this closure should be determined in the ***co-moving (rest) frame*** of the fluid. One again has the option of working in these (generally non-inertial) co-moving frames (so-called Langrangean formulation of the equations) ***or***, of transforming the results back to a fixed inertial Eulerian (i.e., laboratory) frame of reference. As with the radiation field, there are clear advantages and disadvantages to each approach./**6**/

The two dimensionless parameters that sort out the behavior of the electromagnetic fields are *u/c*, and *u/l σ'*. And our limiting regimes of behavior are (throughout we drop all terms of order u^2/c^2):
<u>u/l σ'<<u/c<<1</u>: ***Ideal MHD***: The limit of infinite conductivity ***E'*** is negligible, so ***E*** = - ***u*** x ***B***/c, and the displacement current in Ampère's Law can be dropped. This provides ***J*** in terms of the curl of ***B***. Faraday's Law becomes the magnetic induction equation (without the magnetic diffusion term [6756:2.51]). The Lorentz force takes its familiar form [6756:2.26]. The electric charge density is negligible.
<u>u/c<<u/l σ'<<1</u>: ***Resistive MHD***: ***E'*** is no longer negligible but is essentially solenoidal in character, so we must retain both terms in ***E*** = - ***u*** x ***B***/*c* + ***J***/σ. Faraday's Law is now the full magnetic induction equation provided by [6756:2.9]. The Lorentz force is unchanged.
<u>1<<u/l σ'<<c/u</u>: ***Resistive EHD***: The electric charge density ρ_e' is no longer negligible, and ***E'*** is now irrotational in character. The magnetic field in the commoving frame ***B'*** has become so small (because the solenoidal component of ***J*** has declined at the expense of the irrotational component) that we can approximate ***B*** = + ***u*** x ***E***/*c*. The time derivative in Faraday's Law can be dropped, but the displacement current must now be retained in Ampère's Law. Instead of solving the magnetic induction equation, we now solve the charge conservation equation [6756:2.19] ***but*** with the current density given by ***J*** = σ***E*** + ρ_e***u*** (equivalent to ***J'*** = σ'***E'***)---and ***not*** the expression used in this equation. The electric field ***E*** = ***E'***, is then recovered directly from Gauss's Law, and the Lorentz force is now ρ_e***E***.
<u>1<<c/u<<u/l σ'</u>: ***Ideal EHD***: The conductivity is now so miniscule that the advection of charge dominates the internal currents leading to ***J*** = ρ_e***u*** in the laboratory frame (***J'*** = 0).

In ***MHD***, or Magnetohydrodynamics, the magnetic field dominates over the electric field. The electric current density is solenoidal so there is no electric charge density ρ_e, and therefore the electric field is also solenoidal. Faraday's Law provides the time evolution of the magnetic field. The electric field and the electric current density then follow directly from ***B***. The Lorentz force is dominated by the ***J*** x ***B*** term.

In ***EHD***, or Electrohydrodynamics, the electric field dominates over the magnetic field. The electric current density in the commoving frame is irrotational, so there is almost no magnetic field generated by this current. Instead, through its divergence, it creates an electric charge density. The conservation of electric charge provides the evolution of the electric charge density. The electric field is then obtained from Gauss's Law, and then the electric current density and the magnetic field follow from ρ_e, ***E*** and ***u***. The Lorentz force is dominated by the ρ_e***E*** term.

/**4**/To get the cgs/esu expression given here, begin with the SI result [6756:2.18] and use the equations that relate the tilded and nontilded quantities.

/**5**/Some authors cited above use the negative of our $\mathbb{M}$ as their definition of the Maxwell Stress Tensor. The Lorentz invariance of these equations follow from the Lorentz invariance of the Maxwell Equations. Proving the Lorentz Invariance of the latter can be accomplished in several ways.

/**6**/Both **Mih1984**, **Pom2005**, **Cas2006** and **Hub2015** all have authoritative discussions on this point for radiating fluids. **Rob1967**, **Par1979** and **Dav2001** do the same for magnetized fluids. The earlier works by **Pai1963** and **Pai1966** are also worth consulting. The Lagrangean description is the point of departure for formulating variational approaches for (ideal) MHD and gravity. **Kul2005** and **Ogi2016** both cover the essential aspects of this approach. It is not used much in radiation hydrodynamics because it works best with isentropic (reversible) processes.

The Gravitational Field

It is not straightforward to derive a pair of energy and momentum conservation equations for the gravitational field. The origin of these difficulties traces directly to the fact that there is no time derivative in the Newton/Poisson Equation---the gravitational potential Φ changes immediately everywhere in the universe in response to changes in the matter density. We would have encountered these same issues had we not included the additional time derivative term in the radiation transfer equation. Effecting the same sort of resolution for the gravitational field is possible, but proves to be much more complicated since we must ultimately embed the scalar Φ, into a tensor. The procedure is called the post-Newtonian approximation to general relativity, and it is widely used in many astrophysical applications when a causal treatment of gravity is essential. This approach takes us too far afield. Instead, we will instead finesse our way through some of the ambiguities that arise in leaving the Newton/Poisson Equation as it is.

With these caveats in mind, the starting point is to use our "intuition" to guess that the energy density ***attributable to*** (notice, we are careful here not to say ***of***) the gravitational field is the product of the gravitational potential Φ and the matter density ρ, and we set about computing its time derivative as follows:

$$\frac{\partial}{\partial t}\rho\Phi = \frac{\partial \rho}{\partial t}\Phi + \rho\frac{\partial \Phi}{\partial t}$$

In the second term we replace ρ by the Laplacian of Φ using the Newton/Gauss Equation. Next integrate by parts twice to move the Laplacian from Φ to its time derivative. This generates twice the divergence of the vector $\boldsymbol{G}$. Next exchange the time derivative and the Laplacian and use the Newton/Gauss equation in the other direction to bring back ρ in the time derivative. This creates an additional copy of the first term. Leading to

$$\frac{\partial}{\partial t}\rho\Phi = 2\frac{\partial \rho}{\partial t}\Phi - 2\,\boldsymbol{\nabla}\cdot\boldsymbol{G}$$

The gravitational energy flux $\boldsymbol{G}$ is quadratic in the gravitational potential and contains the antisymmetric product of the gradient of Φ and its time derivative. The gravitational stress tensor $\mathbb{G}$ is:

$$\mathbb{G} \equiv \frac{1}{8\pi G}\left(\|\boldsymbol{\nabla}\Phi\|^2\right)\mathbb{I} - \frac{1}{4\pi G}\left(\boldsymbol{\nabla}\Phi\boldsymbol{\nabla}\Phi\right)$$

A comparison of the expressions for $\mathbb{G}$ and $\mathbb{M}$ suggests that we might instead regard the first term of $\mathbb{G}$ as the energy density of the gravitational field. This too will generate another satisfactory expression if we put a negative sign in front of it and take the energy density of the gravitational field as negative definite. The corresponding flux is then just one of the two terms in our previous expression for $\boldsymbol{G}$.

Clearly the difference between this alternative and our previous expression of the conservation of gravitational energy is an equation of the form

$$\frac{\partial}{\partial t}\Gamma + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = 0$$

Again, it is worth pointing out that the noncausal formulation of gravity is at the heart of many of these issues. The article by **Dam1989** describes the attendant issues of trying to create a causal theory of gravity very clearly without recourse to general relativity. It is an absolutely superb article, and it is well worth your careful study. **Har2001** is also a good place to look. After making your way through this material, you will understand why we didn't try to go there in these two lectures. And also, why, as tempting as it might be to simply turn the Laplacian into the D'Alembertian in the Newton/Poisson Equation, such an approach really does not work.

Ambartsumian & Sobolev

The Matter

We have appended a force density ***f*** on the right side of the equation to account for the exchange of momentum between the material and the (electro)magnetic and radiation fields. One of the contributions to ***f*** is the Lorentz force $\boldsymbol{L}$ you already know, and can immediately write down, from PHY 6756 in the MHD approximation. The contribution from the radiation field will be new, and is determined in the next several slides.

The equation for the internal energy density *e* is (6756:1.59), but we have recast the right side of that equation in a more general form to account for entropy generation associated with the ***irreversible*** transfer of energy between the material and the radiation and (electro)magnetic fields (gravity does not enter here). In any event, the heat conduction by radiation must be omitted here because we shall handle that explicitly in what follows. The thermal conduction by the material can be retained if desired./**7**/

We remark in passing that the internal energy density for the material contains not only the thermal energy due to the random motions of the atoms, electrons, molecules, etc, but also any internal energy reservoirs such as ionization /recombination, excited level populations, population inversions (masers), nuclear reactions, and so forth. Several of these processes were described in Chapter 1 of PHY 3700. Radiation of course plays a very prominent role in mediating many of these atomic and molecular processes, and so it is essential that these processes be treated accurately here on the material side of the ledger as well as on the radiation field side of the ledger. Failure to do so will result in serious errors. Thus, the equation of state ([6756:1.60]), $p = (\gamma-1)\rho e$, $\gamma = c_p/c_V = 5/3$ for a monoatomic gas with no internal degrees of freedom, which omits all of these interesting radiative processes, should be used with ***great care***, if at all in most RMHD applications. With these caveats in mind, we leave the equation of state open for the moment and use the standard manipulations of the three fluid equations (continuity, Euler and internal energy) to arrive at the conservation laws for the kinetic energy density and the internal energy density of the material.

/**7**/I take a lot of liberties with equation (6756:1.59) to arrive at this result. First, the original equation traces its heritage to the thermodynamic relation $de = T\,ds + p\rho^{-2}\,d\rho$ where (again, sorry for the multiple use of symbols), *s* is the specific entropy (entropy per unit mass). **Lan1966** indicates how one goes from thermodynamics to internal energy conservation. In particular, the coefficient of div **u**, is p/ρ---it is $(\gamma-1)e$, ***only*** if it is an ideal gas. The right side internal energy equation is then derived from the *T ds*-term. Thermal conduction, if it is present in any appreciable amount certainly contributes to this term (which deals with irreversible conversion of *ordered* energy to *disordered* energy) and so will the radiation field. In some limiting cases, the coupling of the radiation field to the internal energy can be adequately described by a temperature gradient, but we will derive a result that is more widely applicable later. So I dispense with *both* terms on the right side of [6756:1.59] and replace it with a generic placeholder that will be determined later.

Full Cost Accounting

So, we have succeeded in constructing our set of nine (9) conservation laws. And, in so doing, we have identified the combinations of that describe how energy and momentum are exchanged between the material and the three classical fields. As a bonus, we also can determine which aspects of these changes are irreversible in that they lead to the increase of entropy.

Therefore, it remains only to execute the following tasks, which are required to render this a closed and actionable system of equations.

- [**Not Optional**] We must calculate the integrals that appear on the right sides of the two radiation equations and subsequently determine what contributions they make to the force density and the entropy generation terms in the material equations.
- [**Not Optional**] We must determine two closure relationships to express the radiation pressure tensor $\mathbb{P}$ and the material pressure p in terms of the remaining dynamic quantities.
- [Optional] We may wish to reduce the full Maxwell Equations to the ***magnetohydrodynamic*** (MHD) or ***electrohydrodynamic*** (EHD) limits under the assumption that the electric conductivity σ' in the co-moving rest frame of the material is either immensely large (MHD) or incredibly small (EHD).
- [Optional] We may wish to reduce the radiation hydro equations to the ***diffusion*** or ***streaming*** limits under the assumption that the typical photon mean free path λ is much smaller, or much larger, than any other length scale, l, in the system.

It would take us too far afield, but suffice it to say that energy and linear momentum are not the only useful conservation laws. Angular momentum is quite important, particularly for rotating systems, to treat in parallel with energy and linear momentum. You might like to try to derive an analogous set of conservation of angular momentum equations for the material, radiation, electromagnetic and gravitational fields. Do they tell you anything new? The moments of inertia of a distribution of matter in motion can also yield important relations, the so-called tensor virial equations, that are very helpful in sorting out the behavior of self-gravitating systems. A good place to start with some of these concepts is **Col1978** and references therein.

There is no fundamental reason why isolated magnetic charges (or monopoles, North and South, say) could not exist. No one has found one to date. But they could be accommodated in Maxwell's Equations with a magnetic monopole density ρ_m and corresponding magnetic current density, say $\boldsymbol{J}_m$. On monopoles *per se*, see **Raj2016** and **Gia1984**. For interesting speculations on how Maxwell's Equations and our universe might operate with an ample supply of monopoles see **Par2007**

Symbolically, these equations equate the sum of the *time derivative* of a momentum(energy) density plus the *divergence* of a momentum(energy) flux, with the total (net) *exchange* of momentum(energy) with all the other distinct constituents. When the energy conservation equations for all constituents---fluid kinetic, fluid internal, gravitational, magnetic, and radiation---are added together for an isolated astrophysical system, the exchange terms cancel exactly, leaving a balance between the *time derivative* of the total energy density and the *divergence* of the total energy flux. A similar statement applies for the momentum. Mass, energy, (linear) momentum, angular momentum and certain virial quantities are also conserved for isolated astrophysical systems.

In attempting to arrive at this desired outcome, we shall be forced to grapple with frames of reference and the overarching requirement that our predictions should not depend upon the spacetime coordinate system we employ to carry out our solution of the conservation equations. That this seemingly ancillary complication should arise and play a rather prominent role in our deliberations can be attributed to two essential points.

First, once the material is permitted to move about unfettered in a complicated (perhaps even turbulent!) fashion, one is hard pressed to identify any particular reference frame $\boldsymbol{\Sigma}$ that stands out above all others as the obvious place in which to formulate the problem. Consequently, we are forced to accept the fact that any inertial frame of reference ought to be as good as any other to work in. In other words, we would like the underlying astrophysics to be "invariant" between inertial frames of reference. The transformations of space and time which leave some quantity (or aspect) invariant define the ***geometry*** of that space-time.

Second, in order to be successful with defining the proper geometry it is necessary that the laboratory (inertial reference) frame equations that we used in PHY 3700 and PHY 6756 to define our fields---the fluid equations, Maxwell's Equations, the radiative transfer equation, the magnetic induction equations, the Newton/Poisson Equation---must all be invariant under the same set of transformations. In fact, they are ***not***. The fluid equations, the Newton/Poisson Equation, and the magnetic induction equation are invariant under Galilean transformations between inertial frames. The full Maxwell Equations and the radiative transfer equation are invariant under Lorentz transformations between inertial frames.

There are several strategies for circumventing this inconsistency. We can accept the fact that all experiments to date suggest that the Lorentz transformations (and not the Galilean transformations) are a correct description of the geometry of the spacetime we live in locally (i.e. on scales small compared to the spatial curvature). This means Maxwell and the radiative transfer equation are fine, but we must come up with a Lorentz invariant description of the material fluid equations. And then, the only way to incorporate gravity is through general relativity and the curvature of the space-time. This approach has the advantage that it leaves nothing to the imagination, but the disadvantage that it is very complex and poorly suited to situations in which the typical fluid speeds are much smaller than the speed of light. **Lic1967** and **Syn1957**, are good places to start.

Alternatively, we can work the other direction and try to make suitable modifications of the Maxwell and the transfer equations so that they are Galilean invariant. Indeed, *magnetohydrodynamics* (MHD) as set out Chapter 2 of PHY 6756 is Galilean invariant. In MHD the electric field tends to be solenoidal, and smaller in magnitude than the magnetic field. There exists a second limiting case, called *electrohydrodynamics* (EHD), where Maxwell's Equations can again be made Galilean invariant. In this case the electrical conductivity is assumed to be nearly zero (as opposed to incredibly large, as in MHD), so that there is no electric current in the rest frame of the fluid. In EHD, the magnetic field tends to be smaller in magnitude than the electric field.

But our good fortune ends there. Any efforts to render the radiative transfer equation Galilean invariant does substantial damage to the usefulness of the equation if the radiation is not effectively in the ***static diffusion regime*** (i.e., the photon mean free path λ is much smaller than a typical fluid length scale l, ***and*** the typical fluid speed u is nowhere near the speed of light c). This is because the speed of light is ***not*** a Galilean invariant, and so we make errors in assessing aberration and Doppler shift between moving frames. This in turn leads to spurious energy and momentum losses/gains.

Better Living Through Geometry

An event in the spacetime in which our lives and the laws of physics play out consists of a spatial address provided by three coordinates and a time at which the event takes place at that spatial address. Four numbers---four dimensions to our spacetime. Conversely, spacetime is the collection of all possible spacetime events. Of course, although everyone has their own address systems and watches tailored to suit their needs, for our own sanity, we require that everyone end up with the same physics relating a given sequence of events. Equivalently, we say that they laws of physics should be invariant under mappings or transformations between all ***acceptable*** address systems and time keeping mechanisms. We emphasize the word ***acceptable*** here because we can envision unacceptable systems, for example, a non-inertial reference frames where we must supply additional terms like the Coriolis "force" to obtain invariance (see, for example PHY 6756, Section 1.4).

The so-called ***geometry*** of our 4-dimensional spacetime is determined (a) by those quantities that we require to be invariant, and (b) the collection---or more precisely, the ***group***---of transformations that leave those quantities invariant. In classical, Newtonian physics, this requirement is that the time difference between two distinct spacetime events $|t_1 - t_2|$, ***and***, the spatial (Euclidean) distance between them $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$ should both be invariants. The 10-parameter (Lie) group of linear transformations that guarantee these invariants is called the Galilean Group. It consists of an arbitrary shift of the origin in space and time (4 parameters), a boost to a frame moving at a constant velocity in some arbitrary direction (3 parameters) and a 3-dimensional rotation of the axes of the spatial coordinate system about the origin (3 parameters). Thus, a unique element (transformation of spacetime) of the Galilean Group connects the spacetime coordinates of any two inertial frames of reference. We call this a group because the composition of any two transformations yields a third transformation that is also a member of the group. There is an obvious identity transformation within the group (no rotation, no velocity, no origin translation), and to each transformation there is an inverse that maps the new spacetime coordinates back to the ones you started with./**8**/

To be concrete, let $\boldsymbol{\Sigma'}$, be a frame of reference whose *x-y-z* coordinate axes are rotated by the three familiar Euler angles with respect to a fixed inertial laboratory frame $\boldsymbol{\Sigma}$. Further, suppose that $\boldsymbol{\Sigma'}$ travels at a uniform velocity $\boldsymbol{v}$ with respect to the lab frame $\boldsymbol{\Sigma}$, and that the origins in space and time of $\boldsymbol{\Sigma'}$ and $\boldsymbol{\Sigma}$ are offset by a constant spatial vector $\boldsymbol{b}$ and time τ. Then the elements of the Galilean group that connect these two frames G and G^{-1} are:

$$G(O, \mathbf{v}, \mathbf{b}, \tau)\,[ct,\mathbf{x}]^T = [c(t+\tau), O\mathbf{x}-\mathbf{v}t+\mathbf{b}]^T = [ct',\mathbf{x'}]^T$$

$$G^{-1}(O^T, \mathbf{v'}, \mathbf{b'}, -\tau)\,[ct', \mathbf{x'}]^T = [ct,\mathbf{x}]^T$$

$$\mathbf{v'} = -O^T\mathbf{v}$$

$$\mathbf{b'} = -O^T(\mathbf{b}+\mathbf{v}\tau)$$

Here, O, is a real 3x3 matrix that accounts for the (proper) rotation and depends upon the three Euler angles. It is an element of the Special Orthogonal Group, SO(3): the transpose O^T is also O's inverse, and both of their determinant are +1./**9**/

An ***astrophysicist***, typically arranges matters to dispense with the translation of the spacetime origin (set $\boldsymbol{b}$ = 0, and τ = 0) and uses the proper rotation, O, to conveniently align the boost $\boldsymbol{v}$ with the *x*-axis and *x'*-axis, say, leading to the familiar Galilean transformations:

$$x = x' + vt' \qquad x' = x - vt \qquad y=y' \qquad z=z' \qquad t=t'$$

Conversely, a ***mathematician*** would generalize in quite the opposite direction and notice that the spacetime event can be conveniently described by a 4-vector

$$x = [ct, \mathbf{x}]^T = [x^0, x^1, x^2, x^3]^T = x^\alpha$$

and that the elements of the Galilean Group can then be represented as 4x4 real matrices $\Lambda_\alpha^{\ \beta}$ which act upon the 4-vectors according to:/**10**/

$$x'^\beta = \Lambda_\alpha^{\ \beta} x^\alpha \qquad [\beta = 0, 1, 2, 3 \text{ and summed over all } \alpha\text{'s}].$$

Instead, if we decide to construct our geometry by demanding that the ***proper time interval*** between our two space time events, $-c^2\,|t_1 - t_2|^2 + \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2$, is invariant, then our spacetime is a ***Minkowski*** spacetime. This simple, but conceptually far reaching, change in perspective replaces the 10-parameter Galilean Group, with the 10-parameter Poincaré Group. The Poincaré Group differs from the Galilean Group only in the manner in which the (3 parameter) boosts are treated---the 3-dimensional rotations (encoded in O) and the 4-parameter spacetime origin offset remain unaltered. The mathematician sees no essential difference in structure with the previous situation, the transformation matrix $\Lambda_\alpha^{\ \beta}$ simply contains different entries, that's all. The astrophysicist, of course sees the universe in an entirely different light (pun intended):

$$x = \Gamma_v (x' + vt') \qquad x' = \Gamma_v (x - vt)$$

$$y=y' \qquad z=z'$$

$$t = \Gamma_v (t' + vx'/c^2) \qquad t' = \Gamma_v (t - vx/c^2)$$

$$\Gamma_v = (1 - v^2/c^2)^{-1/2}$$

In the commoving rest frame of the material, a constitutive relation, Ohm's Law, is employed to relate the electric current density to the electric field

$$\mathbf{J'} = \sigma'\mathbf{E'}$$

$$[\,\sigma'\,] = \text{sec}^{-1}$$

In more realistic and complicated situations σ' might be a tensor and the right side of Ohm's Law may also contain the magnetic field $\boldsymbol{B'}$, and pressure gradients. Again, to illustrate how the program works we will take σ' to be a constant scalar. MHD obtains in the limit that the characteristic timescale $1/\sigma'$ is much shorter than any other relevant timescales (like l/u) in the problem under consideration. In other words, there is ample current density to be had for even a modest amount of $\boldsymbol{E'}$. EHD pertains to the opposite case that $1/\sigma'$ is much longer than most relevant time scales of interest in the problem. Put another way, it is very difficult to generate much $\boldsymbol{J'}$ even for whopping electric fields. To determine the Lorentz force back in the Eulerian laboratory frame of reference $\boldsymbol{\Sigma}$, we need the transformation properties of the electromagnetic fields. Although the charge density ρ_e and $\boldsymbol{J}$ combine to form a 4-vector, $[c\rho_e, \boldsymbol{J}]$, which we know how to transform under the Poincaré Group, $\boldsymbol{E}$ and $\boldsymbol{B}$ on the other hand combine to form a 4-tensor, $F^{\alpha\beta}$. ***Mathematically***, tensors transform like

$$F^{\alpha\beta'} = \Lambda_\mu^{\ \alpha}\,\Lambda_\nu^{\ \beta}\,F^{\mu\nu}.$$

Astrophysically, (using the same simplifications discussed above in §VIII) we obtain:

$$E_x' = E_x \qquad B_x' = B_x$$

$$J_y' = J_y \qquad J_z' = J_z$$

$$J_x' = \Gamma_u (J_x - \rho_e u) \qquad J_x = \Gamma_u (J_x' + \rho_e' u)$$

$$\rho_e' = \Gamma_u (\rho_e - uJ_x/c^2) \qquad \rho_e = \Gamma_u (\rho_e' + uJ_x'/c^2)$$

$$E_z' = \Gamma_u (E_z - uB_y/c) \qquad E_z = \Gamma_u (E_z' + uB_y'/c)$$

$$E_y' = \Gamma_u (E_y + uB_z/c) \qquad E_y = \Gamma_u (E_y' - uB_z'/c)$$

$$B_z' = \Gamma_u (B_z + uE_y/c) \qquad B_z = \Gamma_u (B_z' - uE_y'/c)$$

$$B_y' = \Gamma_u (B_y - uE_z/c) \qquad B_y = \Gamma_u (B_y' + uE_z'/c)$$

(Sigh, the ***mathematicians*** have it all over the ***astrophysicists*** when it comes to compact notation.)

/**8**/Those who could care less about transformation groups on spacetimes can skip most of this. If you are happy with a look-up table of transformations between reference frames $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma'}$, then none of this is necessary. For the rest, there is a lot of useful mathematical material on the internet. If you Google subjects like "classical groups", "Galilean Group", "Poincaré Group", "Minkowski Space", "Euclidean Space", "Orthogonal Group" and "Lorentz Group" you have the option of selecting a link that is at the right level. **Pen2005** covers these concepts and provides additional reference material. At times, his approach can be very tough going. The idea that the invariants, and the transformations which preserve these invariants, define the geometry of the space is a profound concept.

/**9**/The elements of orthogonal group O(3) that have determinant -1 still preserve the invariants, but contain reflections which transform right-handed coordinate triads into left-handed triads. So they are discarded in the subgroup SO(3). A subgroup of a group is essentially a self-contained group embedded within the larger group. The representation of the 3x3 orthogonal matrices in terms of the Euler angles is straightforward, but quite cumbersome. **Lan2004** covers this very nicely. SO(3) and the larger Galilean and Poincaré groups are not commutative, so that the order of application of transformations matters. Here, the superscript "T" means transpose.

/**10**/Setting the SO(3) subgroup aside, the Λ's are very easy to write down for Galilean boosts. So easy in fact that one may well wonder why bother with all this machinery. I think it helps to reduce some of the mystery and discomfort around the tensor notation that is employed in relativity. Finally, we are also neglecting the translations in focusing on the matrix representation of the group elements. Both the translations and the proper (determinant = + 1) rotations, while subgroups of the full 10-parameter Lie groups don't add a lot to our knowledge. The boosts are also a subgroup of the Galilean Group but they are ***not*** a subgroup of Poincaré Group: the composition of two nonparallel boosts induces a rotation related to the (Lllewellyn) Thomas Precession.

Lorentz Transformations of the Radiation

In the commoving frame of the fluid, ***Σ'***, one can, of course, based on your work in PHY 3700, devise all sorts of fascinating and complicated behaviors for the opacity $\chi'_{\nu'}$ and the emissivity $\eta'_{\nu'}$ and their dependences upon ***n'***. To illustrate how RMHD works, we'll take the simplest illustrative example, and leave the remainder for you to explore. We'll treat the atmosphere as grey, and omit any scattering ($\chi'_{\nu'} = \kappa$, κ a constant). The source function is just the Planck Function based on the temperature of the material./**11**/ As you discovered in your work on stellar atmospheres, these are in fact not bad approximations to reality in a number of astrophysical circumstances. In the inertial, laboratory frame, ***Σ***, however, the emission and the opacity are ***not*** isotropic due to the Doppler shift and aberration of the photons between these two frames.

The energy and momentum of a photon of frequency *ν* and propagation direction ***n*** can be combined into a (Galilean or Lorentz, take your pick if you are of the mathematician persuasion) 4-vector, $h\nu c^{-1}$ [*1*, ***n***] and transform the same way as the spacetime event [*ct*, ***x***] depending upon which spacetime geometry you prefer. We'll leave the easy (Galilean) one to you. For the astrophysicist, (with the boost ***V*** aligned with the *x*-axis and *x'*-axis as before),/**12**/

$\nu = \Gamma_V \nu' (1 + V n'_x/c)$ $\quad$ $\nu' = \Gamma_V \nu (1 - V n_x/c)$

$n_y = (\nu'/\nu) n'_y$ $\quad$ $n_z = (\nu'/\nu) n'_z$

$n_x = (\nu'/\nu)\{n'_x + \Gamma_V (V/c)[1 + (\Gamma_V V n'_x/c)/(\Gamma_V + 1)]\}$

$n'_x = (\nu/\nu')\{n_x - \Gamma_V (V/c)[1 - (\Gamma_V V n_x/c)/(\Gamma_V + 1)]\}$

Setting the Lorentz factor $\Gamma_V = 1$, and retaining only the first-order terms in *V/c* provides the usual formulae for the Doppler shift and aberration between moving frames of reference. Of course, you found the Galilean version of these to be remarkably simple---so simple in fact that it does not even provide the frequency Doppler shift! And so, it seems rather pointless to press on hoping for a Galilean invariant formulation for the radiation field.

/**11**/The definition of the Planck Function, $B_\nu(T)$, is provided by equation (3700:2.4.8).

/**12**/I am using a capital "*V*" here because the nu "*ν*" is so hard to distinguish from the lower case vee "*v*". Sorry about this font silliness. At any rate, it is easy enough to derive these equations for yourself noting that nu "*ν*" transforms like *ct*, and that *ν**n*** transforms like ***x***. Or, you can look them up in **Mih1984**.

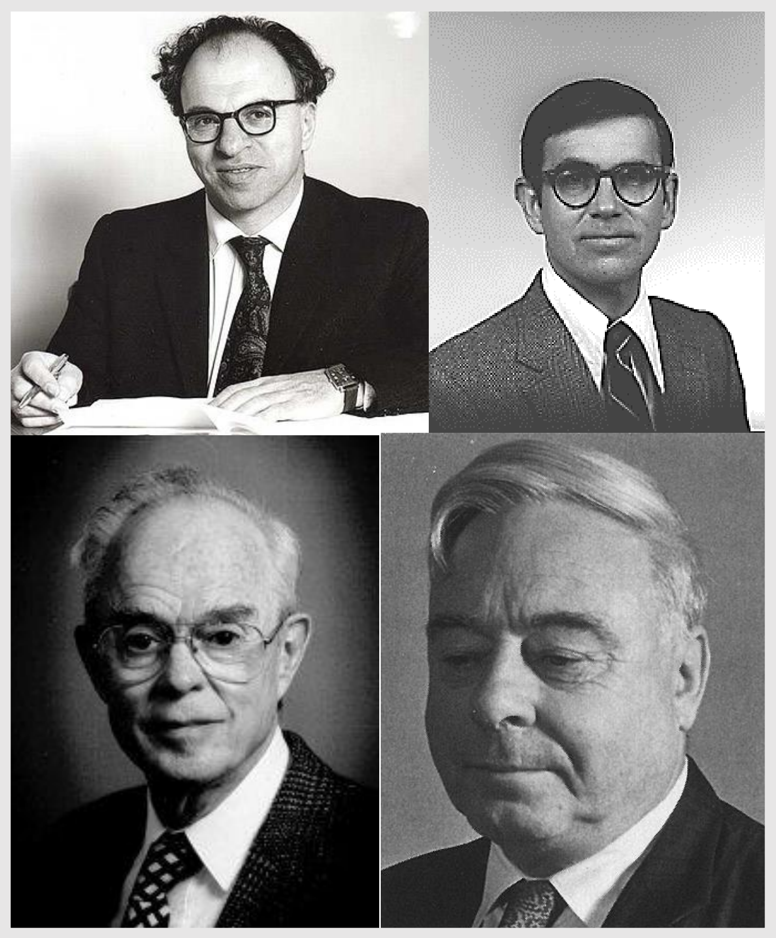
/**13**/The Appendix of **Pom2005** takes you through the steps.

Now, if we transform the differential operator on the left side of the transfer equation using the transformation properties of {*t*, ***x***, ***n***} to {*t'*, ***x'***, ***n'***} we find after some algebra that we end up with exactly the same differential operator in the commoving frame ***except*** that it is multiplied by an overall factor of (ν'/ν). /**13**/ Since both sides of the transfer equation must transform the same way to be invariant, we immediately know that the opacity has to transform in the same fashion as the differential operator! So this gives us one of the transformation properties presented on this slide.

To figure out how the emissivity transforms we ultimately have to figure out how the specific intensity transforms. This is a much more complicated endeavor. First, observe that we actually know how the photon phase space volume element $d\nu d\mathbf{n}$ transforms to $d\nu' d\mathbf{n}'$ because we know how the frequency and direction of propagation transform. Next, we note that the number density of photons is the specific intensity divided by $h\nu c$. This is the number per unit volume, per Hz, per steradian. Because we know how the spacetime volume element $dt d\mathbf{x}$ transforms we can now put all of this together and arrive at an expression for how many photons each observer ***counts***. In order that they agree on this number we can then determine how the specific intensity must transform to assure this outcome. **Mih1984** gives a very thorough discussion on this point as does **Syn1957**. The essential argument is due to Llewellyn Thomas in 1930. Once we know how the specific intensity transforms, the invariance of the transfer equation yields the emissivity transformation properties.

It is worth mentioning here that the radiation field can also be ***polarized***. In many cases this polarization, be it circular, linear or some combination of the two, represents a small percentage of the overall intensity. However, as detectors and telescopes improve it has become possible to measure very small levels of polarization from a variety of astrophysical objects. The monograph **Lan2004** is ***the*** reference for the transfer of polarized radiation. The essential approach is to replace the specific intensity, which is a scalar function, with a vector containing four quantities (including the specific intensity as one entry) that take into account the state of polarization of the radiation. This vector is called the Stokes Vector. The transfer equation becomes a matrix equation for the four components of the Stokes vector, taking into account the possibility that the four components are mixed in passing through matter. In particular, the presence of magnetic and electric fields in the material are effective agents from inducing polarization. Therefore the measurement of polarized radiation from astrophysical objects offers the tantalizing prospect of deducing critical information about the nature of the magnetic, and possibly electric, fields.

Bondi Mihalas



Parker Thomas

Spherically-Symmetric Winds & Accretion

Mathematically, the 9 conservation laws take the form of nonlinear partial differential equations in four independent variables (3 space + 1 time). And generally speaking, with the exception of the tools and methods provided by the Lie Theory of symmetry transformations, as described for example by **Ste1989** or **Can2002**, they must be solved by numerical methods and approximation schemes. The one exception to this prevailing situation involves the use of special coordinate systems that are in some sense tailored to the geometry of the problem, be it a star (spherical, oblate or prolate coordinates), a double star system (bispherical coordinates), or a gaseous disk of some sort (toroidal, flat-ring cyclide coordinates). **Moo1988** is a treasure trove of everything you could want to know about the available options and their essential properties.

This zoo of coordinate systems can be particularly useful because it sometimes is possible to separate the PDEs into some number of ODEs and remaining PDEs of lower dimensionality. The monographs by **Cam1997** and **Bes2010** do this to great advantage for double stars and disks. As a rule, ODEs are much easier to solve numerically or analytically than PDEs. In the best of all possible situations, embodied by the cylindrical and spherical coordinate systems, the metric scale factors that enter into the definitions of div, grad and curl depend only upon the single (radial) coordinate. Thus a class of solutions exist that depend upon only a single spatial coordinate. And further, if we seek steady solutions, we drop the time derivatives and our PDEs are truly just ODEs. In this lecture we restrict our attention to spherical systems to reap the benefits of these simplifications.

The integration constants provided by the conservation of mass and energy, while very helpful in facilitating and interpreting a solution of the steady spherical accretion and wind problems, do not actually yield the complete solution to the problem. This unfortunate outcome is related to the fact that while the total energy of the flow is conserved, the flow does in general find it expeditious to transfer energy between kinetic, internal, gravitational, and radiation fields during the course of its sojourn from the stellar surface to the interstellar medium or vice versa. The net conservation statement cannot tell us why, how and where these transformations take place. The exchange terms that were derived in Lecture 1, of course, are what is required to describe these conversions. It follows that we need to keep one, or maybe two, of the individual conservation laws in play. Because we have the overall conserved integration constant, one of these equations follows directly from the solution of the others. In rotating (e.g. axisymmetric) systems, the conservation of angular momentum will yield an additional integration constant, but at the cost of an additional dependent variable (i.e., the azimuthal fluid velocity, say) and an additional PDE. Chapter 3 of PHY6756 provides a nice illustration of this point.

It is a very nice fact that the steady Euler Equation (and its Navier Stokes counterpart) are invariant under changing the sign (e.g., direction) of the radial velocity. So we get one equation that can describe both stellar winds and accretion onto compact objects! In this lecture you should always think about both applications in parallel.

Here, we shall also assume that the flow resides around a massive condensed central object whose properties do not vary over the times scales of interest. This will be a reasonable approximation if the mass loss or gain is sufficiently small. This is realized in many, but not all, astrophysical situations. Now you can do something really fun. Try to put the time dependence back into the problem, and use the isothermal wind solution, say, to determine how the mass of the central object decays with time. In so doing, you will want to ensure that the gravitational potential weakens appropriately as the mass loss shrinks the central object. How long until our sun, for example, disappears? [Hint: use some of the typical solar values provided in Chapter 3 of PHY6756.]

Likewise, we shall assume that the gravitational field is produced entirely by the central compact object, and that the surrounding flow is sufficiently tenuous that its self-gravity can be neglected. Again, this is not unreasonable in many cases. Mathematically, however, it poses no essential difficulty to include the self-gravity of wind, although it does certainly complicate the exposition. You might wish to take the isothermal wind solution and compute the self-gravity of the material in the wind a posteriori and see how large it is compared to the gravity of the central object. If you do so, you will find a puzzling result. What does that result suggest? For fun, you might try to write down, and see if you can solve the isothermal wind with self-gravity taken into account. What assumptions will you need to make to ensure that the problem is well-posed? Does self-gravity increase or decrease mass loss, terminal wind speed?

The isothermal solution is mathematically singular in the sense that you lose equation **II** of the Bondi/Parker Equations, and therefore to compensate for that, you also lose the energy flux integral! Were this not the case, the problem would become ill-posed. In a certain limiting sense this corresponds to the idea that thermal conduction becomes so efficient in the wind that no temperature gradient is allowed to exist anywhere. You might like to see how this limit comes about by using equation (6756:1.59) and keeping only the thermal conduction term on the right side of this equation and modifying equation **II** to incorporate thermal conduction. How far can you push the analysis? If you run into a wall take a peak at **Lam1999**.

The Bondi-Parker Equations, labelled **I** and **II**, date back to the 1950's. It is curious that the accretion aspect emerged first and the wind aspect almost a decade later. Perhaps thermodynamics has something to do with it. The accretion flow takes a very organized radial flow and converts it into disorganized thermal energy which, from an entropy perspective seems to go in the right direction. The wind, on the other hand, takes disorganized thermal energy and converts it into an organized radial outflow. Thinking about this rather blithely, it would seem to violate the second law of thermodynamics. Of course, it does not actually do so. Your refrigerator would also violate the second law of thermodynamics if you used this same reasoning---you can of course decrease the entropy if you compensate by doing *pdV* work on the system! This is what Parker's wind, and your refrigerator, both manage to do.

Yet, even prominent astrophysicists sometimes get confused by in the second law of thermodynamics. Joseph Chamberlain asserted that only the Class I solutions were consistent with the physics and the Parker solution was unphysical. Indeed, Parker was denied tenure at the University of Chicago largely because of the accumulating criticism around his singular solution. Parker told me that John Simpson, a cosmic ray physicist on the Chicago faculty, stood up to the physics department and insisted that he and his team would leave if Parker was denied tenure. Simpson was sufficiently persuasive to cause the faculty to reverse their decision. Parker stayed. And the supersonic solar wind as Parker predicted was confirmed a few years later by spacecraft measurements.

Nowadays one can find a number of papers where various individuals claim that the supersonic wind solution was actually "discovered" much earlier by people like Sydney Chapman, Ludwig Biermann. Paul Ahnert and Cuno Hoffmeister. History always looks quite different from the perspective of the future.

From an accretion perspective, the Class I solutions are also physically admissible. The conversion of gravitational energy to thermal energy keeps ahead of the conversion to kinetic energy throughout the flow, setting up a pressure gradient that decelerates the inflow.

If you perturb the steady Parker and Bondi solutiions, you find that the former is stable while the latter is unstable to the formation of a shock front that precipitously takes the flow from supersonic to subsonic.

Radiatively Driven Winds

In spherical geometry, the transfer equation takes the following form, where μ is the cosine of the angle between the unit vector in the r-direction and the photon propagation direction $\mathbf{n}$.

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t}+\mu\frac{\partial I_\nu}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu}=\eta_\nu-\chi_\nu I_\nu$$

The purpose of the additional term on the left side of this equation can be understood by a simple geometrical construction. On a piece of paper place a dot to represent the origin of the spherical coordinate system and then draw any straight line that does not pass through the origin to represent the trajectory of a photon. Where the photon makes its closest approach to the origin, the value of μ is zero, and as the photon is located farther and farther from the origin μ approaches plus or minus one. In fact, with a little effort it should be possible to write down a simple equation that determines the value of μ in terms of the radial distance of the photon from the origin and the impact parameter at closest approach. Indeed, the left side of the transfer equation, should be such that it leaves this relationship unaltered for a free-streaming photon.

Mathematically, this suggests that we look for a new set of coordinates say (ξ, ϖ) to replace (r, μ), where one of the coordinates, say ϖ, accounts for the conserved relationship between r and μ that you discovered from your geometrical drawing. The combination

$$\varpi=r\sqrt{1-\mu^2} \qquad \xi=\frac{r}{\sqrt{1-\mu^2}}$$

does the trick. If you now use the chain rule to work out the left side of the transfer equation in these new coordinates you will find that there is no derivative with respect to ϖ. So the transfer equation contains only the single derivative with respect to ξ and can be treated by the same methods you used in PHY3700 to solve the one-dimensional slab problems! This approach is called the method of characteristics, and is discussed in **Can2002**, for example. Several of the references on radiative transfer cited in the bibliography handle the spherical geometry explicitly. Ambartsumian is credited with developing this transformation.

Using the (ξ, ϖ) coordinates, you might find it amusing to go back to Chapitre 2 of PHY3700 and carry out some of the calculations presented in one-dimensional slab geometry now in one-dimensional spherical geometry!

For pure Thomson Scattering, for example, all terms on the right side of the energy conservation equation for the radiation field are of order u^2/c^2. This remarkable fact, for a steady radiating flow, asserts that the radiation flux $\boldsymbol{F}$ truly falls off ***exactly*** as one over r^2. Such is ***not*** the case in the gray atmosphere approximation where differences between the specific intensity of the radiation field and the Planck Function result in the transfer of energy from the radiation to the material or vice versa.

This state of affairs for pure scattering may, at first sight, seem somewhat bizarre. After all, if the radiation field is acting to drive the material outward by scattering off the free electrons (or decelerate it for material falling inward), shouldn't it be transferring energy to the material as these electrons pull the protons and other positive ions along with them? Indeed, such would be the case in a time-dependent problem. But steady state calculations have provided the radiation field, and all the other components of the problem, an infinite amount of time to settle down. The photons scattered isotropically by the electrons in their rest frame have had all the time necessary to go rattling around throughout the flow and communicate upstream and downstream conditions throughout the entire flow.

You may find it instructive to carry through the same calculations for the gray atmosphere approximation omitting any scattering effects to understand where some of these distinctions arise.

Line-driven winds are treated by **Lam1999**, and also by **Mih1984**, and **Hub2015**. **Sob1960** is also worth a look. A careful treatment of the theory requires much more time and effort than we can possibly devote to it here. The essential idea is that hot stars of early spectral type emit a lot of ultra-violet radiation and abundant ions of carbon, nitrogen and oxygen, have a plethora of UV transitions that can be very effective in absorbing the UV continuum radiation from the star. So effective, indeed, that if the atmosphere was static these lines would be optically thick. However, as the wind begins to move away from the star, the ions in the rest frame of the material see radiation from the surface of the star Doppler shifted to the red from line center. Hence there are fresh photons available to absorb and deposit momentum in the flow. This causes the wind to accelerate and access continuum photons further to the red of line center. This continues to accelerate the wind until the material encounters the "shadow" of a neighboring UV line.

The phenomenological form of the line factor can then be understood as an expression of the acceleration of the material as reckoned in the co-moving frame! The greater the acceleration, the more effectively the C, N, O ions can access continuum photons to the red of the absorption below them. For an ensemble of lines that overlap we get a reduction in the efficiency of the process by the power of α determined by Castor, Abbott and Klein in their seminal paper on the subject. The P Cygni line profile is the characteristic spectral signature of such a process in operation.

Blast Waves

The fact that the material equations do not contain any conspicuous length scales or time scales, suggests that homologous expansions and contractions ought to preserve the structure of the equations. In other words, power law scalings may leave the equations invariant. As we learned earlier from our discussion of spacetime geometry, invariants preserved by transformations are incredibly powerful in unravelling the behavior of systems and providing clues on how to derive solutions of the equations. This, in fact, is the essential idea behind the Lie Theory of continuous groups and their applications to nonlinear ODEs and PDEs. Here again, **Ste1989**, and **Can2002** are your first stop to learn more about these powerful methods. We arrived at these self-similar solutions by noodling around, but the true power of the Lie Theory is that there is no noodling around---it tells you precisely ***how*** to find the symmetries (although, it may be exceedingly difficult to follow its directions).

As the little image on the slide suggests, the Sedov/Taylor/von Neumann problem was motivated, to some extent, by trying to understand what was to be expected when a nuclear bomb was set off in the atmosphere. Hence gravitational effects are unimportant to leading order, as is the density and pressure stratification of the atmosphere. They can be accounted for a posteriori by a perturbation analysis if so desired.

Sedov and his school, were fond of calling these symmetry solutions 'intermediate asymptotics'. By this they meant that nature settles into the self-similar solution some time ***after*** the initial release of a huge amount of energy, but ***before*** the blast wave has overrun so much material that it is beginning to exhaust that initial supply of energy in plowing up the surrounding material. Their contention is that in this intermediate phase, the flow has forgotten almost everything about the initial conditions except how much energy it was given, and could care less about what it is overrunning. It therefore settles into a universal state that is independent of the specific details of the problem.

The essential point is that the jump conditions in density, velocity and pressure at the shock front must have the same time scaling as the interior post shock flow. This in turn requires that the inflow Mach number of the material must be very small. After a sufficient amount of time, when this is no longer the case, the solution must depart from the self-similar form.

Two interesting books that serve as a jumping-off point for these sorts of investigations, and their applications to supernovae for example, are **Arn1996** and **Kor1991**.

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