In topology, a topological space is called simply-connected (or 1-connected) if it is path-connected and every path between two points can be continuously transformed, staying within the space, into any other such path while preserving the two endpoints in question (see below for an informal discussion).

If a space is not simply-connected, it is convenient to measure the extent to which it fails to be simply-connected; this is done by the fundamental group. Intuitively, the fundamental group measures how the holes behave on a space; if there are no holes, the fundamental group is trivial — equivalently, the space is simply connected.

1 Informal discussion

Informally, a thick object in our space is simply-connected if it consists of one piece and does not have any “holes” that pass all the way through it. For example, neither a doughnut nor a coffee cup (with handle) is simply connected, but a hollow rubber ball is simply connected. In two dimensions, a circle is not simply-connected, but a disk and a line are. Spaces that are connected but not simply connected are called non–simply-connected or, in a somewhat old-fashioned term, multiply connected.

A sphere is simply connected because every loop can be contracted (on the surface) to a point.

To illustrate the notion of simple connectedness, suppose we are considering an object in three dimensions: for example, an object in the shape of a box, a doughnut, or a corkscrew. Think of the object as a strangely shaped aquarium full of water, with rigid sides. Now think of a diver who takes a long piece of string and trails it through the water inside the aquarium, in whatever way he pleases, and then joins the two ends of the string to form a closed loop. Now the loop begins to contract on itself, getting smaller and smaller. (Assume that the loop magically knows the best way to contract, and won’t get snagged on jagged edges if it can possibly avoid them.) If the loop can always shrink all the way to a point, then the aquarium’s interior is simply connected. If sometimes the loop gets caught — for example, around the central hole in the doughnut — then the object is not simply-connected.

Notice that the definition only rules out “handle-shaped” holes. A sphere (or, equivalently, a rubber ball with a hollow center) is simply connected, because any loop on the surface of a sphere can contract to a point, even though it has a “hole” in the hollow center. The stronger condition, that the object has no holes of any dimension, is called contractibility.

2 Formal definition and equivalent formulations

A topological space \( X \) is called simply-connected if it is path-connected and any continuous map \( f : S^1 \to X \) (where \( S^1 \) denotes the unit circle in Euclidean 2-space) can be contracted to a point in the following sense: there exists a continuous map \( F : D^2 \to X \) (where \( D^2 \) denotes the unit disk in Euclidean 2-space) such that \( F \) restricted to \( S^1 \) is \( f \).

An equivalent formulation is this: \( X \) is simply-connected if and only if it is path-connected, and whenever \( p : [0,1] \to X \) and \( q : [0,1] \to X \) are two paths (i.e.: continuous maps) with the same start and endpoint \( (p(0) = q(0) \) and \( p(1) = q(1)) \), then \( p \) and \( q \) are homotopic relative \( [0,1] \). Intuitively, this means that \( p \) can be “continuously deformed” to get \( q \) while keeping the endpoints fixed. Hence the term simply connected: for any two given points in \( X \), there is one and “essentially” only one path connecting them.

A third way to express the same: \( X \) is simply-connected if and only if \( X \) is path-connected and the fundamental
group of $X$ at each of its points is trivial, i.e. consists only of the identity element.

Yet another formulation is often used in complex analysis: an open subset $X$ of $\mathbb{C}$ is simply-connected if and only if both $X$ and its complement in the Riemann sphere are connected.

The set of complex numbers with imaginary part strictly greater than zero and less than one, furnishes an nice example of an unbounded, connected, open subset of the plane whose complement is not connected. It might also be worth pointing out that a relaxation of the requirement that $X$ be connected leadsto an interesting exploration of open subsets of the plane with connected extended complement. For example, a (not necessarily connected) open set has connected extended complement exactly when each of its connected components are simply-connected.

### 3 Examples

- A torus is not simply connected. Neither of the colored loops can be contracted to a point without leaving the surface.

- The Euclidean plane $\mathbb{R}^2$ is simply connected, but $\mathbb{R}^2$ minus the origin $(0,0)$ is not. If $n > 2$, then both $\mathbb{R}^n$ and $\mathbb{R}^n$ minus the origin are simply connected.

- Analogously: the $n$-dimensional sphere $S^n$ is simply connected if and only if $n \geq 2$.

- Every convex subset of $\mathbb{R}^n$ is simply connected.

- A torus, the (elliptic) cylinder, the Möbius strip, the projective plane and the Klein bottle are not simply connected.

- Every topological vector space is simply-connected; this includes Banach spaces and Hilbert spaces.

- For $n \geq 2$, the special orthogonal group $SO(n,\mathbb{R})$ is not simply-connected and the special unitary group $SU(n)$ is simply-connected.

- The long line $L$ is simply-connected, but its compactification, the extended long line $L^*$ is not (since it is not even path connected).

- Similarly, the one-point compactification of $\mathbb{R}$ is not simply-connected (even though $\mathbb{R}$ is simply-connected).

### 4 Properties

A surface (two-dimensional topological manifold) is simply-connected if and only if it is connected and its genus is 0. Intuitively, the genus is the number of “handles” of the surface.

If a space $X$ is not simply-connected, one can often rectify this defect by using its universal cover, a simply-connected space which maps to $X$ in a particularly nice way.

If $X$ and $Y$ are homotopy-equivalent and $X$ is simply-connected, then so is $Y$.

Note that the image of a simply-connected set under a continuous function need not be simply-connected. Take for example the complex plane under the exponential map: the image is $\mathbb{C} - \{0\}$, which clearly is not simply connected.

The notion of simple connectedness is important in complex analysis because of the following facts:

- If $U$ is a simply-connected open subset of the complex plane $\mathbb{C}$, and $f : U \rightarrow \mathbb{C}$ is a holomorphic function, then $f$ has an antiderivative $F$ on $U$, and the value of every line integral in $U$ with integrand $f$ depends only on the end points $u$ and $v$ of the path, and can be computed as $F(v) - F(u)$. The integral thus does not depend on the particular path connecting $u$ and $v$.

- The Riemann mapping theorem states that any non-empty open simply connected subset of $\mathbb{C}$ (except for $\mathbb{C}$ itself) is conformally equivalent to the unit disk.

The notion of simple connectedness is also a crucial condition in the Poincaré lemma.
5 See also

- Deformation retract
- n-connected
- Path-connected
- Unicoherent

6 References

7 Text and image sources, contributors, and licenses

7.1 Text


7.2 Images

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