Comparison of Very High Prandtl Number and Infinite Prandtl Number Plumes Using the Adjoint Equations

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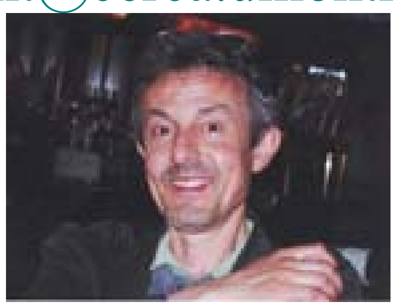

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Introduction

- The numerical models of plumes grown from a point heat source using the finite Prandtl equations are compared with those grown using infinite Prandtl approximations.

- Plumes are grown for Prandtl numbers up to 2×10^4 and Rayleigh numbers up to 10^8. Plumes grown at high finite Prandtl numbers are compared to plumes grown using the infinite Prandtl number approximations.

- The adjoint method is being developed to more accurately calculate the growth of point source plumes.

Finite Prandtl Number Equations

Nondimensionalized by the freefall velocity [1],

$$U = \sqrt{\alpha \Delta T g d}$$

Conservation of Mass:

$$\nabla \bullet \mathbf{v} = 0$$

Conservation of Momentum:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \omega + T\mathbf{e}_z - \nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{v}$$

Conservation of Energy:

$$\frac{\partial T}{\partial t} = -\mathbf{v} \bullet \nabla T + \frac{1}{\sqrt{PrRa}} \nabla^2 T$$

Infinite Prandtl Number Equations

Nondimensionalized by the thermal diffusivity

Conservation of Mass:

$$\nabla \bullet \mathbf{v} = 0$$

Conservation of Momentum:

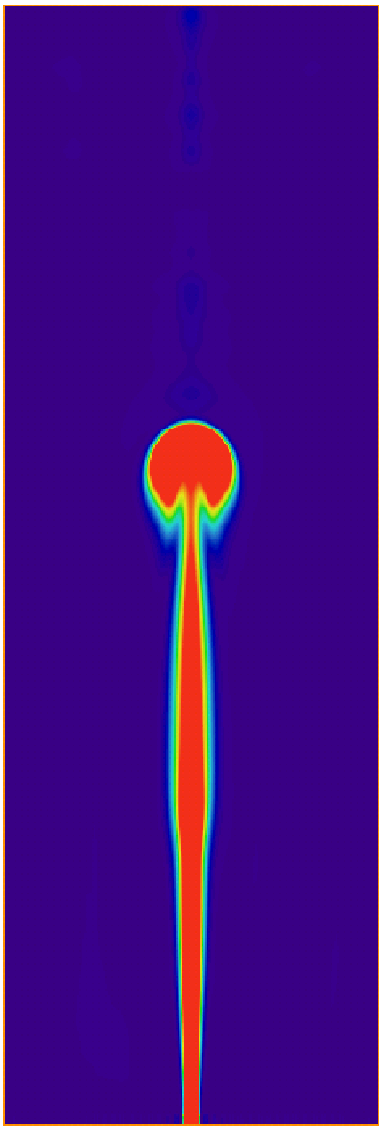
$$Ra T \mathbf{e}_z + \nabla^2 \mathbf{v} - \nabla p = 0$$

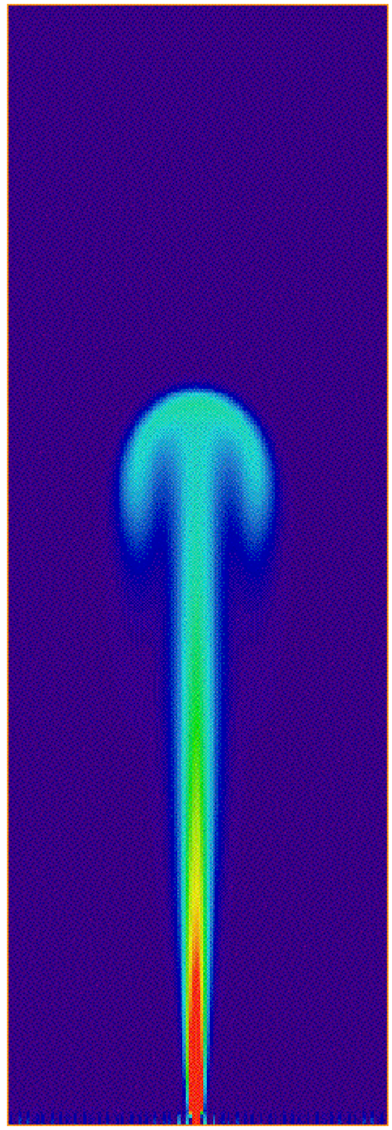
Conservation of Energy:

$$\frac{DT}{Dt} = \nabla^2 T$$

$Ra = 10^6$

Plumes are compared at similar heights.
(see movies on computer)





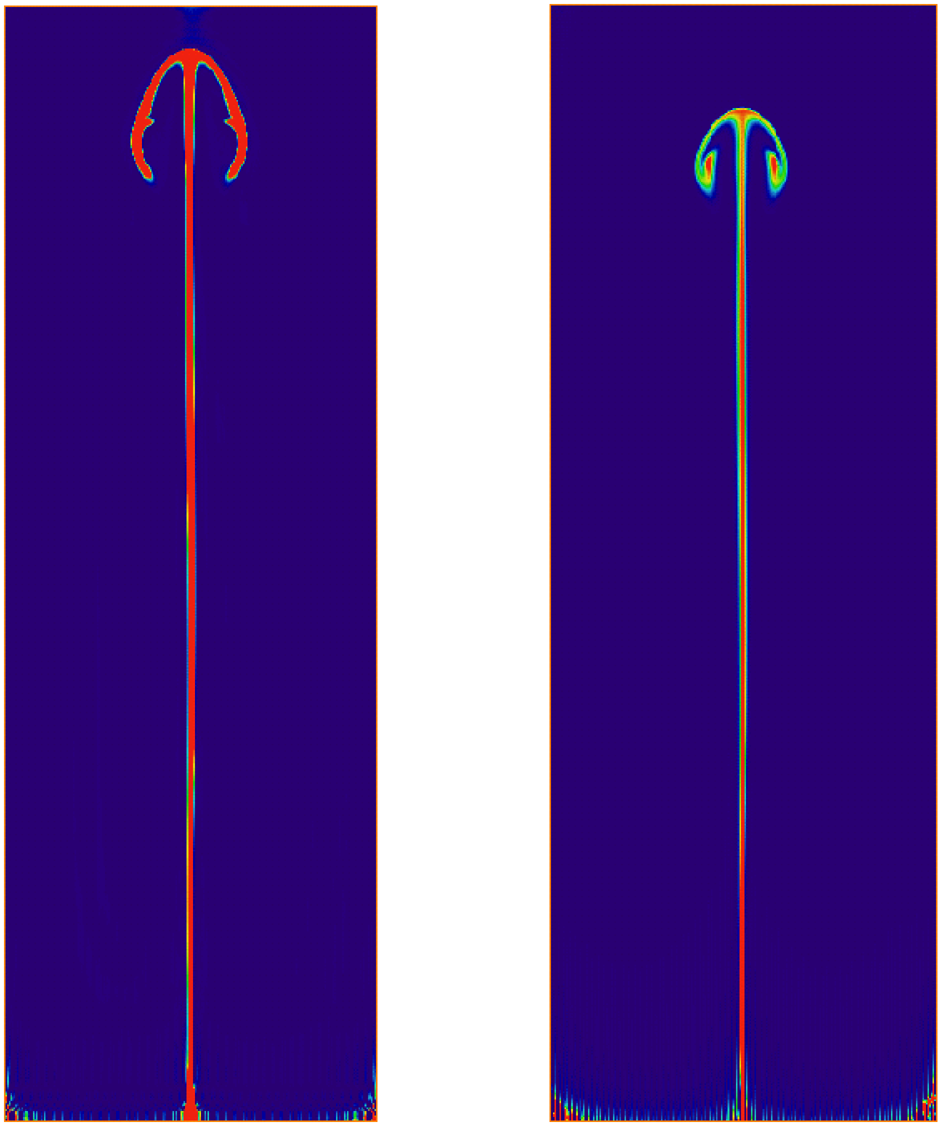
$Pr = 10^4$ $t = 0.000782$

$Pr = \text{infinity}$ $t = 0.00225$

*all times are nondimensionalized by the thermal diffusivity

$Ra = 10^8$

Plumes are compared at similar heights.
(see movies on computer)



$Pr = 2x10^4$
$t = 1.55x10^{-5}$

Pr =infinity
$t = 5x10^{-4}$

*all times are nondimensionalized by the thermal diffusivity

Results Without 4D-VAR Method

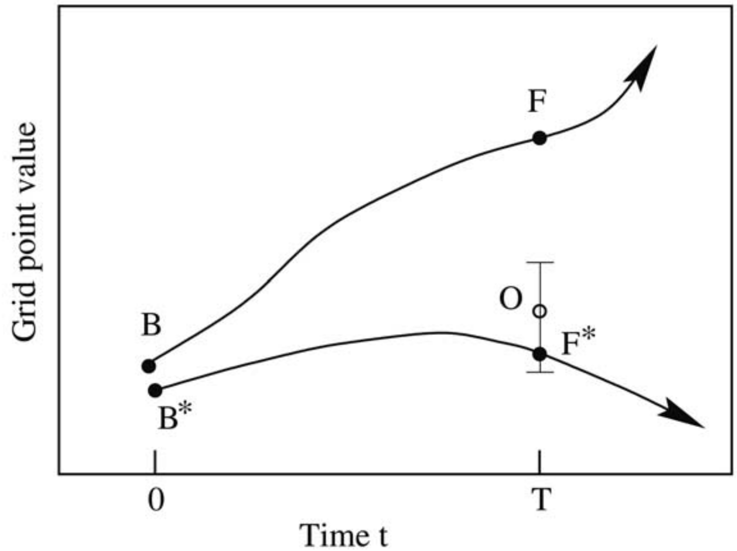
- Plume growth for finite Prandtl and infinite Prandtl plumes is significant even at Prandtl numbers on the order of 10^4.
- At lower Rayleigh numbers (10^6), finite Prandtl number plumes with Prandtl number 10^4 are much more thermally diffusive than infinite Prandtl plumes.
- At higher Rayleigh numbers (10^8) finite Prandtl number plumes with Prandtl number $2x10^4$ still have inertial structures, such as waves, that are not seen in infinite Prandtl number approximations.
- Important implications for numerical models of large, but finite Prandtl fluids, such as magma and mushy ice, that are often approximated by the infinite Prandtl number solution [2,3].

Adjoint Equations and the 4D (space and time) variational method

- One method to obtain better forecasts is to correct erroneous initial conditions by minimizing the error between the forecast and the observations [4,5]. This method, known as 4D-VAR, has been used previously in forecasting the behavior of rivers and dams during floods (Bélanger et al 2002, see computer).

- Adjoint integration gives the sensitivity of the physical fields to the initial conditions, boundary conditions and other parameters [6].

Illustration of 4D-VAR method [7]



B is the original initial conditions
F is the original forecast
O is the observations
B* is the corrected initial conditions
F* is the corrected forecast

4D-VAR Plume Study

- We will begin with an initial plume simulation. This simulation will be used as the “observations” since we do not have any experimental observations available.

- We will then slightly alter the temperature in the domain, which alters the initial heat flux. The simulation run with the altered temperature will be the “predictions”.

- We then minimize the difference in the heat flux between the “observed” and “predicted” simulation with a minimization algorithm, such as steepest descent.

Cost Function

The cost function J is given by:

$$\mathrm{J} = \int_{\Omega} \int_{0}^{t_{final}} \mathbf{f}\left(\vec{\Psi}, \vec{x}, t\right) d\vec{x}\, dt$$

where f represents the error between the predictions and observations, and $\vec{\Psi}$ is the system state variables, such as temperature and velocity [8,5].

In this study, the cost function is expressed as a function of the heat flux, $\mathrm{H}\left(\vec{\Psi}\right)$:

$$\mathrm{J} = \frac{1}{2} \sum_{i=1}^{N} (\mathrm{H}_i - \mathrm{H}_i^{obs})^2$$

The Variational Problem

We need to find the time trajectory that minimizes the cost function while satisfying the constraints given by the physical equations, $\mathcal{E}(\vec{\Psi},\vec{x},t)=0$. This is done with the Lagrangian:

$$\mathcal{L}(\vec{\Psi},\vec{\lambda}) = \mathrm{J}(\vec{\Psi}) + \int_{\Omega} \int_{0}^{t_{final}} \vec{\lambda}(\vec{x},t)\cdot\mathcal{E}(\vec{\Psi},\vec{x},t)\,d\vec{x}\,dt$$

where $\vec{\lambda}$ are the Lagrange undetermined multipliers or adjoint variables [9]. The minimum of the Lagrangian gives the minimum of the cost function since $\mathcal{E}(\vec{\Psi},\vec{x},t)=0$ [8].

The Lagrangian reaches a minimum when [10]:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \vec{\Psi}} \cdot \delta \vec{\Psi} + \frac{\partial \mathcal{L}}{\partial \vec{\lambda}} \cdot \delta \vec{\lambda} = 0$$

The minimum of the Lagrangian is found when [6]:

$$\frac{\partial \mathcal{L}}{\partial \vec{\lambda}} = \mathcal{E}(\vec{\Psi}, \vec{x}, t) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \vec{\Psi}} = \mathrm{Adj}(\vec{\lambda}) + \frac{\partial \mathrm{J}}{\partial \vec{\Psi}} = 0$$

The two above equations are the Euler-Lagrange equations [11].

Adjoint Equations for Finite Prandtl Convection

Conservation of Mass:

$$\nabla^2 P^* = \frac{\partial v_x^*}{\partial x} + \frac{\partial v_z^*}{\partial z} - \frac{\partial J}{\partial P}$$

Conservation of Momentum in x:

$$\frac{\partial v_x^*}{\partial \tau} = v_x \frac{\partial v_x^*}{\partial x} + v_z \frac{\partial v_x^*}{\partial z} + \frac{1}{\sqrt{RaPr}} \nabla^2 v_x^* - v_x \frac{\partial v_z^*}{\partial z}$$

$$+ v_z \frac{\partial v_z^*}{\partial x} + v_x \nabla^2 P^* - 2 v_x \frac{\partial^2 P^*}{\partial x^2} - 2 v_z \frac{\partial^2 P^*}{\partial x \partial z} + T \frac{\partial T^*}{\partial x}$$

Conservation of Momentum in *z*:

$$\frac{\partial v_z^*}{\partial \tau} = v_z \frac{\partial v_z^*}{\partial z} + v_x \frac{\partial v_z^*}{\partial x} + \frac{1}{\sqrt{RaPr}} \nabla^2 v_z^* - v_z \frac{\partial v_z^*}{\partial x}$$

$$+ v_x \frac{\partial v_x^*}{\partial z} + v_z \nabla^2 P^* - 2 v_z \frac{\partial^2 P^*}{\partial z^2} - 2 v_z \frac{\partial^2 P^*}{\partial x \partial z} + T \frac{\partial T^*}{\partial z} - \frac{\partial J}{\partial v_z}$$

Conservation of Energy:

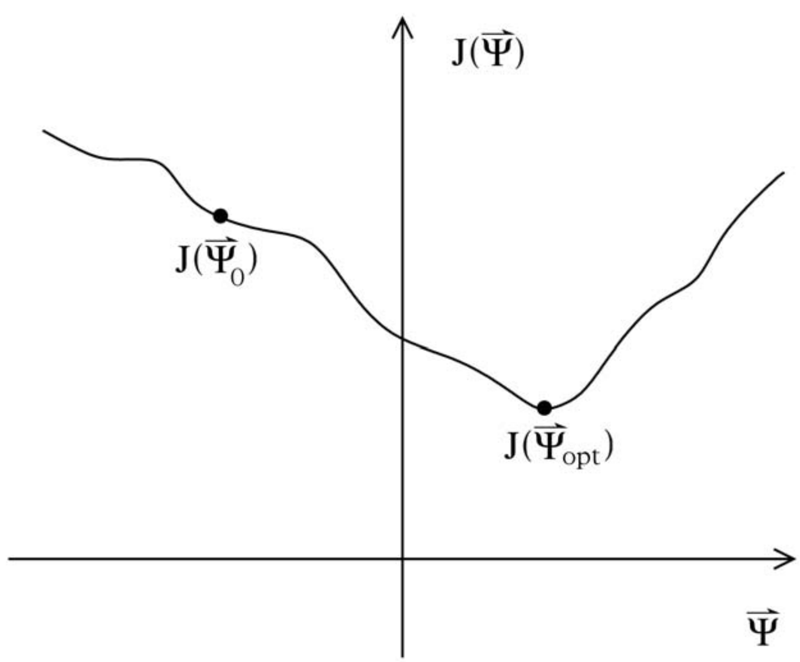
$$\frac{\partial T^*}{\partial \tau} = \mathbf{v} \bullet \nabla T + \sqrt{\frac{Pr}{Ra}} \nabla^2 T^* + v_z^* - \frac{\partial P^*}{\partial z} - \frac{\partial J}{\partial T}$$

* indicates Lagrange undetermined multiplier (or adjoint variable)

τ is the inverse time $(\tau = t_{final} - t)$

Minimization of the Cost Function

We want to find the optimal initial conditions, $\vec{\Psi}_{opt}$, that minimizes the cost function [9].



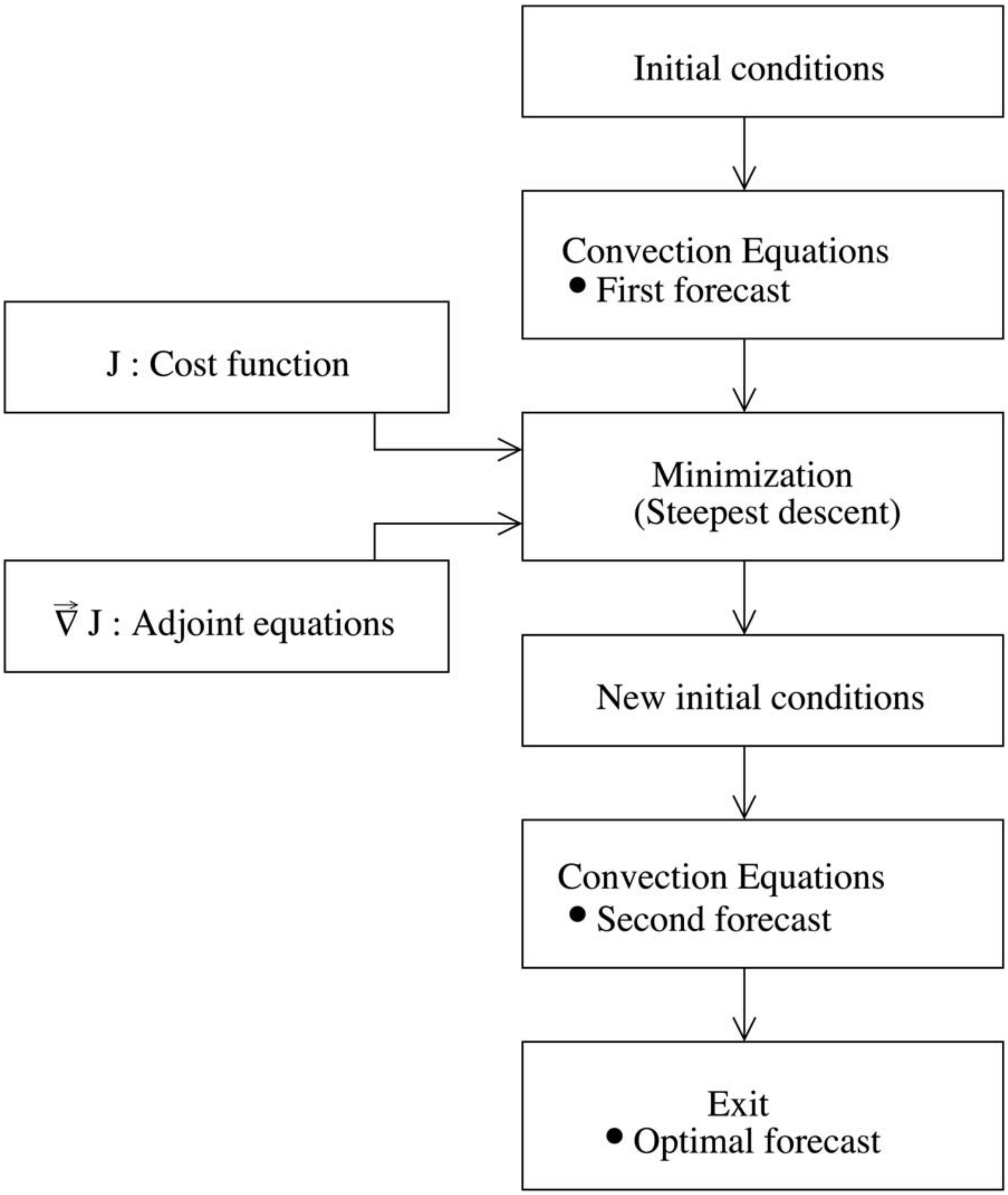
For the steepest descent algorithm, we first evaluate the cost function with an initial guess. We then find the gradient of J with respect to the initial conditions from [5]:

$$\nabla_{\vec{\Psi}_0} J = \vec{\lambda}\Big|_{\tau=0}$$

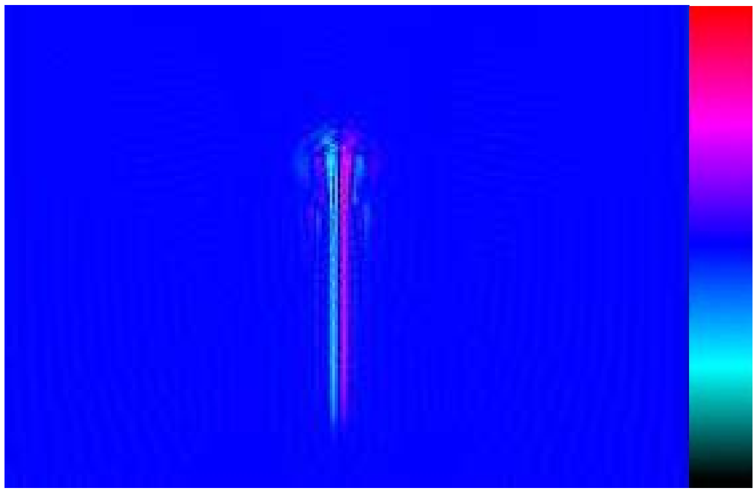
We next move in the opposite sense of the gradient J. We continue until $\nabla J \approx 0$, which gives us the optimal initial conditions [12].

It is the minimization of the cost function which requires the majority of the CPU time since the adjoint and forward integration must be repeated until the minimum is reached. It takes about 15-20x more CPU time to use the 4D-VAR method than to solve the conservation equations directly.

Algorithm


Initial conditions
Convection Equations
• First forecast
J : Cost function
Minimization
(Steepest descent)
$\vec{\nabla}$ J : Adjoint equations
New initial conditions
Convection Equations
• Second forecast
Exit
• Optimal forecast

Sensitivity for Temperature



∇J_{T_0} for a plume with $Ra = 10^8$, $Pr = 10^4$ at $t = 5.6\text{x}10^{-7}$, where t is the thermal diffusive time. The color scale ranges from $-8\text{x}10^{-4}$ to $8\text{x}10^{-4}$. The plume has reached 1/4 of the box height.

Adjoint Integration versus Backward Integration

- Backward integration involves the evolution of the physical field back in time; whereas for adjoint integration it is the sensitivity fields that are integrated back in time. The sensitivity fields are the partial derivatives of a cost function, J, with respect to the initial conditions, boundary conditions or other parameters [5,6].

- Backward integration is unstable when diffusive terms are present. For example, temperature cannot diffuse from cold to hot. Thus, backward in time integration can be problematic for low Rayleigh number processes where diffusion is important. Since a minus sign arises in the adjoint equations, the integration of diffusive terms is well-posed [5].

- Even though adjoint and backward integration can both be used to achieve the same goal in some cases, the adjoint integration is more proper and rigorous [5].

Conclusions and Future Work

- We have derived the adjoint equations of the convection equations.

- We will use these equations and the 4D-VAR method to gain a better understanding of the dependence of finite Prandtl plume growth on the initial conditions. We plan to study how the small changes in the temperature field can affect the solution.

- This will give us a better idea of how sensitive the finite Prandtl plume simulations are to the initial temperature condition. This information will allow us to better address how significant the differences are between the finite and infinite Prandtl simulations.

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